DECOMPOSITION METHODS FOR MANAGING SERVICE PARTS WITH COUPLED DEMANDS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Collin S. K. Chan

August 2012

DECOMPOSITION METHODS FOR MANAGING SERVICE PARTS WITH COUPLED DEMANDS

Collin S. K. Chan, Ph.D.

Cornell University 2012

We consider the inventory management problem of procurement and allocation for a non-stationary equipment overhaul problem with stochastic demands for multiple job types requiring different combinations of service parts over a finite horizon. Because the service parts involved tend to be expensive, inventory needs to be managed carefully in order to operate in a cost-effective manner. This problem is complex because the procurement lead times tend to be long. We incorporate into our model holding costs for unused service parts and backordering costs for outstanding jobs. Using linear programs to approximate the value function which appears in the dynamic programming formulation, we derive a non-standard allocation procedure which is neither first-come-first-served nor myopic. To obtain procurement decisions, we decompose the original problem into multiple single-part-multi-job-type sub-problems using dual variables. These sub-problems may be tackled by another layer of decomposition using the classical result of Clark and Scarf (1960). Using independent order-up-to procurement and myopic allocation as the bench mark, our policy performs better for the canonical "N" system with two service parts and two job types and also for larger problems.

In the second part of this dissertation, we focus on a single-part-single-job-type problem where backorders accumulate increasing per-period backlogging charges. We show that the state space may still be collapsed into the single-

dimensional inventory position using a non-traditional dynamic program formulation where the immediate cost function consists of all the expected holding and backordering charges associated with the procured units. We use stopping-time random variables to capture the periods in which the procured units are matched up with customer requests. Using this alternate cost accounting mechanism, we independently show the optimality of base-stock policies, a result also obtained by Huh et al. (2011) using a more traditional approach. This dynamic program is suitable for computation and it allows us to compute the optimal base-stock levels.

Finally, we extend these results to two-echelon inventory distribution systems with aging backorders. Because the first-come-first-served allocation of inventory is not necessarily optimal in a distribution network, we modify our cost-accounting mechanism at the level of the upper echelon and derive a dynamic program which is a lower bound for the exact problem. This lower-bound dynamic program has the advantage of using a state vector that does not need to distinguish among backorders of different ages. We decompose this lower-bound dynamic program using Clark and Scarf's (1960) approach to derive yet another lower bound consisting only of single-location problems with single-dimensional state variables. A numerical study is carried out to see how the performance of an operating policy driven by this decomposition compares with the lower bound obtained.

BIOGRAPHICAL SKETCH

Collin Sik Kin Chan was born on January 10, 1983 in Hong Kong and lived there until moving to Greater Vancouver, BC in Canada at the age of thirteen. Upon the completion of his undergraduate studies in Mathematics and Physics at the University of British Columbia in 2006, he began his doctoral studies in Operations Research at Cornell University. Collin put his faith in Jesus Christ while studying at Cornell and was baptized on August 30, 2009 at the First Ithaca Chinese Christian Church. Upon finishing his studies at Cornell in July 2012, Collin will join the research group at Bank of America Merrill Lynch in New York City. Collin is excited to venture into industry and will keep his eyes open for opportunities to work as an educator which he feels passionate about.

To my parents

ACKNOWLEDGEMENTS

My undergraduate calculus of variations teacher once mentioned that the ability to optimize even in infinite-dimensional spaces made him feel powerful. This was part of my mindset when I started my studies in a field that strives to optimize decision making under uncertainty. Ironically, of all the things that happened over the last few years at Cornell, what stands out the most in retrospect is an OCD episode during which I became powerless in dealing with some of the most mundane uncertainties in life. This may sound a bit unfortunate, but I look back humbly with gratitude and am glad to have made it through with all the people I am blessed to be surrounded by.

It has been an honour to work with my advisors Professor Peter Jackson and Professor Huseyin Topaloglu and I thank them for making the completion of this dissertation possible. I thank my advisors for being patient with my shortcomings. Their mathematical wit, economical insights and dedication to work as researchers and educators have been inspirational. Professionally, my advisors have mentored me with a peculiar balance of supportive encouragement and frank advice. At a more personal level, I thank them for making me feel comfortable and welcome in talking to them at all times. This was especially invaluable during the tougher period a few years ago. In addition to my advisors, I want to thank Professor John Muckstadt for giving me an interesting and well-defined research problem to work on. I thank Professor Muckstadt for treating us as his colleagues and I am grateful for the advice he has given me over the years. For serving on my thesis committee, I thank Professor David Shmoys. For reminding me his availability when my other advisors were on-leave, I thank him for being approachable.

Aside from my committee, I thank the many wonderful teachers I have had at Cornell. During my time at Cornell, I have also had multiple opportunities to serve as an instructor or a teaching assistant. I thank everyone whom I have had the opportunity to work with in teaching. Financially, I have been supported in the past six years by a Cornell fellowship, the Shum award, an instructorship, multiple teaching and research assistantships as well as the NSERC Type M and Type D-3 Grants and I thank all of their providers. I would also like to thank all the administrative staff at ORIE, especially Kathryn King, for all the support they have provided.

I cannot be where I am today without my family. I do not take for granted this opportunity to pursue my own interests and I thank my parents and sister very much for their continual support and encouragement. I thank my parents for the many years of upbringing. To the many brothers and sisters at church whom I have been able to confide my struggles in, I thank you for your continual prayer support. To the many friends I have at Cornell and in Ithaca, I thank you for making Ithaca such a wonderful place which I will always miss. It has been wonderful having Kathleen King and Baldur Magnusson as my officemates and friends since my first year at Cornell. Together with Jie Chen, Tia Sondjaja and Gwen Spencer, they have made coming to the office a great joy. To King Yin Wong, Matt McLean, Yuemeng Sun, and Wallace Hui, I express my utmost gratitude for what you have done for me during the most difficult times. There must have been much annoyance and pain in putting up with my anxiety-driven and selfish requests which are often ridiculous. I also wish to thank all the staff and health care professionals at Gannett for taking care of my health all these years at Cornell.

Finally, I thank You for finding me when I was lost and for being the all in all from whom all blessings flow. Thank You for drawing me close to You.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Our interest in this problem originated in a project by Yap et al. (2005) and Al-Gwaiz et al. (2006) with the aerospace unit of Honeywell. In order to more consistently attain the service levels prescribed in its commitment to its customers, Honeywell wanted to reduce its turnaround time of a typical job repair. The inventory control problem faced by Honeywell Aerospace is made more difficult by the fact that such a complex piece of equipment as a jet engine could require many different combinations of service parts to repair. At the time the problem was taken on, the optimization was carried out at Honeywell assuming that the demands for the different service parts were independent. Without taking the correlated demands into account, one neglects situations when the presence of a service part may be of no use at all unless coupled with another part. Another layer of complexity is added by the competition for the same parts by demands and backorders that differ not only in the subset of service parts required, but also in the level of urgency. Using as their objectives the maximization of fill-rate as well as the minimization of average waiting time, Yap et al. (2005) and Al-Gwaiz et al. (2006) proposed methods that incorporate correlations and found cases in which these methods outperformed algorithms with assumptions of independence.

One of the challenges involved in studying this type of equipment overhaul problem is how we should model the correlated demand processes of the different service parts using historical data. An interesting and difficult problem in its own right, this statistical problem was tackled using clustering analysis by Al-Gwaiz et al. (2006). As a result of such analysis, we may view the demands for service parts in the system as being triggered by a set of job types each having their distinct bill-of-material requirements, levels of urgency as well as arrival patterns over time. The schematic diagram below illustrates this modeling approach. It also gives the reader some idea of the difficulty of the equipment overhaul problem.

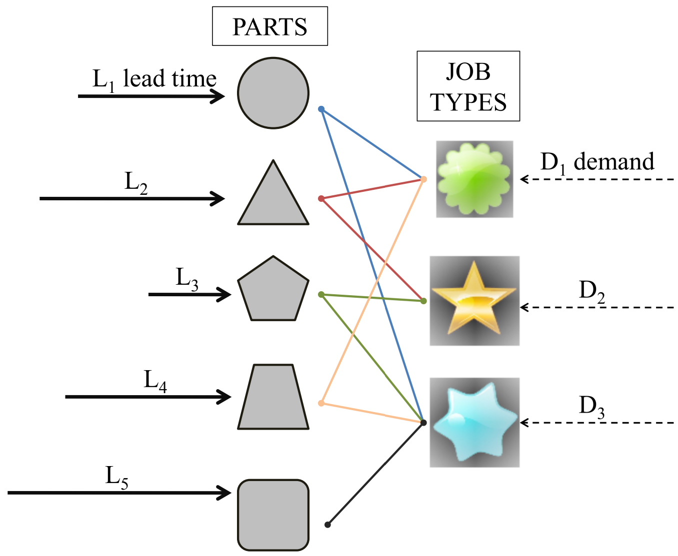

Figure 1.1: This figure illustrates an equipment overhaul system with 5 service parts and 3 job types. The service parts have associated with them different procurement lead times and the demand processes for the three job types are different. The connections between the service parts and the job types illustrate the bill-of-material requirements.

In this dissertation, we focus on the sequential decision-making problem by assuming that all of these parameters have been provided. The goal of our research is to derive computationally-tractable algorithms which can be applied to manage the logistics involved in these problems in a manner that is more effective than first-come-first-served or myopic allocation and independent procurement. Our solution approach makes use of a stochastic dynamic program whose objective is to minimize the total expected holding and backorder costs incurred over a finite horizon. While the holding cost of inventory can be estimated by considering its storage and maintenance costs, the backorder cost of unfulfilled demands is a management parameter which can be adjusted in practice until desirable fill-rates are obtained. The minimization of total holding and backorder cost is therefore linked to the minimization of such performance measures as fill-rates and average waiting times. At each decision point, we determine how on-hand inventory should be allocated among existing backorders and how much inventory should be purchased in expectation of future demands. A challenge of the equipment overhaul problem is the interplay between allocation and procurement decisions.

Discrete-time stochastic dynamic programming is a very useful tool for analyzing sequential decision-making problems under uncertainty. We start in each period of the planning horizon with some information pertaining to the state of the system. Taking into account this information and our expectation on what might happen in future periods, a decision is made and the system undergoes some random events and transitions to the next period. Given a suitable terminal boundary function $V_{T+1}(\cdot)$, the Bellman's equation below can be used to find the optimal policy which may be used to operate the system assuming that the objective is to minimize the total cost incurred over the finite horizon:

$$V_t(x_t) = \min_{a_t \in \mathcal{A}_t(x_t)} \left\{ \mathbb{E}[C_t(x_t, a_t, W_t)] + \mathbb{E}[V_{t+1}\left(x_{t+1}(x_t, a_t, W_t)\right)] \right\}. \tag{1.1}$$

The function $V_t(x_t)$ is called a value function and it captures the minimum total expected cost incurred over the planning horizon starting in period t if we employ the optimal policy starting in period t in state x_t. The set $\mathcal{A}_t(x_t)$ consists of all the actions (decisions) we can take (make) given that we are in state x_t at the beginning of period t. Our decision a_t has both short-term and long-term cost effects. Depending on the outcome of the random event W_t that is to take place between period t and period $t+1$, we incur a one-period cost of $C_t(x_t, a_t, W_t)$ while transitioning to period $t+1$. The $\mathbb{E}[V_{t+1}\left(x_{t+1}(x_t, a_t, W_t)\right)]$ part of (1.1) captures the long-term impact of our decision a_t in the form of the total expected cost incurred under the optimal policy starting at $t+1$ in state $x_{t+1}(x_t, a_t, W_t)$ which in turn depends not only on x_t and W_t but also on the decision a_t. This part of the equation is also known as the cost-to-go function.

By formulating the equipment overhaul problem as a finite-horizon cost minimization problem with sequential decision-making, one could theoretically solve the Bellman's equation using backward recursion. Unfortunately, for a problem of this type where the state vector keeps track of the on-hand, and on-order inventory of every service part in addition to the number of backorders of each job type, and the action space consists of all possible combinations of allocation and procurement decisions, the number of states to consider explodes in size for all but trivial problems. The approach of backward recursion becomes quickly intractable. Therefore, one must resort to approximation ideas to efficiently tackle this problem, while not relaxing it to the point of losing the essence of the demand correlation structure.

We start with a first-order approximation of the cost-to-go function appearing in the Bellman's equation. Whereas a myopic allocation policy completely ignores the value of holding inventory until the next period and the hidden cost of carrying backorders to the next period, we estimate them using deterministic linear programs with sampled future demands. This approximation leads to an improved allocation policy. Unfortunately, a first-order approximation of the cost-to-go function renders procurement decisions indeterminate in the sense that the relaxed Bellman's equation where the cost-to-go function is replaced by its linear approximation does not tell us how much inventory should be purchased when it is desirable to place an order because the linear approximation ignores the decreasing marginal benefits of extra inventory in the system.

In solving the procurement part of the equipment overhaul problem, which consists of multiple job types and multiple service parts, we use a decomposition technique to break it up into many single-part-multi-job-type problems. Resembling a distribution network where demands occur for the different job types but an inventory of service parts is kept centrally, these single-part-multi-job-type problems are solved using another layer of decomposition following the work of Clark and Scarf (1960). This layer of decomposition breaks the single-part-multi-job-type problems into multiple single-part problems.

The chapters of this dissertation are organized as follows. Chapter 2 formulates the equipment overhaul problem with linear holding and backorder costs, and outlines our proposed allocation policy using a first-order approximation of the value function appearing in the Bellman's equation. It then considers a non-linear approximation of the value function which allows us to separate the original problem into multiple single-part-multi-job type problems. We describe how Clark-and-Scarf's decomposition can be applied to solve these subproblems. Numerical results are presented to show the advantage of our policy over ones that make use only of myopic allocation and independent procurement.

In contractual repair settings where customers impose service penalties for failing to meet promised repair times such as in the equipment overhaul problem, a linear cost model which ignores the ages of outstanding backorders could become insufficient. So, in Chapter 3, we consider an alternate cost structure where backorders accumulate increasing per-period backorder costs as they age. In the context of a single-location single-item problem, as shown in Huh et al. (2011), we characterize the optimal policy as a base-stock policy but we use a different cost-accounting mechanism from that used by Huh et al. (2011). This cost-accounting mechanism considers the matching of procured units with future demands at the time of purchase. Using this same cost-accounting mechanism, we extend the analysis to a two-echelon inventory distribution system with aging backorders in Chapter 4. While the optimality of base-stock policies cannot be proven, we propose a computationally tractable algorithm for solving this new problem also using Clark-and-Scarf's approach which results in multiple dynamic programs with single-dimensional state spaces. The results of our short numerical study are presented in Chapter 3 and in Chapter 4. The results of Chapter 4 can be applied to solve the single-part-multi-job-type subproblems which arise in a version of the equipment overhaul problem where increasing per-period costs are used for aging backorders.

The chapters in this dissertation are written in a manner which generally allows them to be read independently of one another. But with Chapter 4 being an extension of the work presented in Chapter 3, the reader will be guided to refer to results found in Chapter 3 where needed.

CHAPTER 2

AN OVERALL DECOMPOSITION APPROACH FOR THE SERVICE PARTS MANAGEMENT PROBLEM

2.1 Chapter Abstract

We consider the problem of procuring and allocating service parts for a non-stationary equipment overhaul problem with stochastic and coupled demands for service parts due to the presence of multiple job types requiring different combinations of service parts. We incorporate into our model holding costs for unused service parts and backordering costs for outstanding jobs. In practice, for tractability, one would assume that parts in such a system are procured independently of one another and according to an order-up-to procurement policy. Similarly, when parts must be allocated among competing uses, one might practically assume that a myopic allocation rule would be used. Accordingly, we set as a benchmark for our study the use of independent order-up-to procurement policies and myopic allocation policies. Because of the correlated nature of the demands, it is reasonable to expect that such policies, though tractable, may significantly fall short of more coordinated policies. Therefore, we formulate the problem as a dynamic program without any a priori restriction on the form of the procurement and allocation policies. In order to make the resulting dynamic program solvable in a computationally tractable manner, we make a linear approximation of the value function using deterministic linear programs. This leads to a price-driven allocation policy that takes into account the hidden cost of using on-hand inventory which is neglected in a first-come-first-served policy typically assumed in the literature. For procurement decisions, we use

decomposition to break up the original problem into many single-service-part subproblems. This leads to a non-linear approximation of the value function which is tighter than the linear approximation. To evaluate the effectiveness of these value function approximations, we focus first on a simple system consisting of two service parts and two job types, in which one of the service parts is required by both job types. For this so-called "N" structure, we identify conditions under which the value function approximation approach dominates the traditional benchmark approach. We then repeat the study for larger systems.

2.2 Introduction

In this chapter, we consider an equipment overhaul problem in which coupled demands are observed for the service parts which are required to repair arriving jobs. For example, several parts in the same module in a jet engine are often replaced together. Because the service parts involved tend to be expensive, inventory needs to be managed carefully in order to operate in a cost-effective manner. A good inventory policy should strive to balance the cost of holding expensive service parts against the cost of delaying job completions. This problem is complicated by the fact that the time it takes to procure the service parts tends to be long compared to the time it takes to diagnose and actually repair a job. We model this inventory planning problem by introducing purchasing, holding and backordering costs. We aim to minimize the total cost over a finite planning horizon.

We consider a periodic-review model in which allocation and procurement decisions are made in each period. Each service part has associated with it a known constant procurement lead time. Holding costs are incurred for unused on-hand components after allocation decisions are made. While the exact dynamic program can be formulated easily, backward induction cannot be used in practice due to the sizes of the state and action spaces. We propose to estimate the value functions appearing in the dynamic program formulation using deterministic linear programs. Solutions to these linear programs allow us to estimate the value of being in a state. We use the dual prices obtained to drive our allocation decisions. While this approach yields good allocation decisions, these prices unfortunately render procurement decisions indeterminate in that purchase quantities would be set either at zero or infinity. Consequently, to determine a procurement policy, we further decompose the problem into multiple single-part-multi-job-type problems with adjusted backordering costs. We use Clark and Scarf's approach (Clark and Scarf (1960)) to solve these sub-problems in order to get procurement quantities.

The contributions of this chapter include the application of a linear-program-based decomposition approach to equipment overhaul problems as well as the derivation of an alternate algorithm which is computationally tractable. We test this algorithm on a simple so-called "N" system and some larger systems and find many cases in which the extra computational effort of this approach yields superior results to the simple benchmark approach.

The rest of this chapter is organized as follows. The next section reviews each relevant area of research. We describe the mathematical model in section 2.4. In section 2.5, we outline the deterministic and randomized linear programs underlying the optimal control problem and discuss their use in deriving an allocation policy. The dual variables from these mathematical programs lead naturally to the decomposition method which we propose to use for the procurement part of the problem. This procedure is outlined in section 2.6. The decomposition reduces the original problem into single-part-multi-job-type problems and we describe in section 2.7 how they may be solved using the classical approach outlined by Clark and Scarf (1960). Numerical experiments were run to test the performance of our proposed inventory policy. Their results are summarized in section 2.8 and the chapter is concluded with further remarks in the last section.

2.3 Literature Review

The equipment overhaul literature is concerned with supply chains involving service parts. Of special relevance here is research focused on planning at the tactical and operational levels. Together with Nahmias (1981), Guide et al. (1997) and Kennedy et al. (2002), Muckstadt (2005) gives a good overview of up-to-date research on the analysis and algorithms used for service parts supply chains. At the tactical level, the interest is in the levels of inventory required at each entity of the supply chain to reach a certain performance level. It is often the case that such optimization is carried out assuming that the system is in a steady state. At the operational level, the interest is in decisions made in real-time pertaining to how a job should be responded to based on the state of the system. An area that awaits further exploration in this literature is where demands for different service parts are correlated. The recent work of Vliegen (2009) considers the integrated planning for service tools and spare parts for capital goods in a service parts supply chain network. Approximation methods are outlined for various performance measures including fill rates. Base-stock optimization is carried out based on these estimations. We focus in this chapter on a one-location problem and treat both the procurement and allocation of service parts as real-time decisions that depend on the state of the system, as opposed to tactical planning decisions (Muckstadt (2005)) which are considered in Vliegen (2009).

A literature which is highly related to our problem is that of assemble-to-order (ATO) systems. Such systems are characterized by the stocking of inventory only at the part level. These parts are assembled into final products for which random future demands arise. The time it takes to assemble these final products is considered negligible which explains why stocking is done only at the part level. Assemble-to-order systems are difficult to analyze and optimize due to the need of coordinating procurement and allocation decisions for different parts. Song and Zipkin (2003) provide an excellent overview of assemble-to-order systems. Another related problem is that of job-kit optimization. An example of such work can be found in Mamer and Smith (2001). We now focus on the subset of work in the ATO literature that aligns with the choices we make in modeling our system. Two special cases of an assemble-to-order system are worth mentioning: a distribution system which consists of one single part required by multiple job types and an assembly system which consists of multiple parts but just one job type. Clark and Scarf's seminal paper (Clark and Scarf (1960)) solves a serial system to optimality and demonstrates that their approach would also solve the distribution problem to optimality if certain balance assumptions hold. This approach in general gives rise to lower bounds for the value functions appearing in the exact dynamic program formulations. Federgruen and Zipkin (1984) give an alternate approach that relaxes the non-negative shipment constraint of each of the downstream-locations. This method is further refined by Kunnumkal and Topaloglu (2008) by introducing Lagrangian variables for these non-negative shipment constraints. Also worth mentioning here is Rosling (1989) who successfully reduces a pure assembly system into an equivalent serial system with properly adjusted lead times and echelon inventory positions. With a long-run balance assumption satisfied, the optimal inventory policy for such an assembly system is characterized completely.

A general assemble-to-order system with arbitrary parameters does not have a characterized optimal solution. The recent work of Dogru et al. (2010) show an optimal policy for an ATO system with three parts and two job types (a "W" system) where the supply lead times of the service parts are all deterministic and identical. Using a two-stage stochastic program with recourse to bound the value function appearing in the problem formulation from below, a policy is derived that attains this lower bound and hence for the first time, an optimal policy is obtained for a non-trivial assemble-to-order system. Departing from the restrictive assumption of equal lead times, the problem complicates very quickly. Much of the work in the literature, therefore, pertains to certain subclasses of inventory policies and is focused on the estimation of performance measures or the (approximate) optimization over such subclasses of policies.

In continuous-review models, some form of first-come-first-served (FCFS) allocation and base-stock procurement are typically assumed. It is generally the case that arrivals follow some form of Poisson process. Optimization problems considered in such contexts involve performance measures such as order fill rate and average number of backorders which themselves need to be estimated. Some works focus on these estimation techniques such as Song (1998), Song et al. (1999), Song (2000), Song (2002) and Lu et al. (2003). In a similar spirit, Glasserman and Wang (1998) study the tradeoff between inventory investment and the length of the time window within which customer demands for final products are satisfied at various fixed fill-rates (the number of demands satisfied within the time window). When the fill rate is high, the tradeoff is linear and a way of computing this rate of compensation is proposed. These estimation problems are themselves difficult.

Given estimates and bounds for the fill rate and the average number of backorders, Lu et al. (2005), Song and Yao (2002) consider constrained optimization problems involving these measures in their constraints (e.g. some minimum fill rate needs to be satisfied) or objectives (e.g. the minimization of a weighted average number of backorders). In Lu and Song (2005), an unconstrained optimization problem is solved to optimality assuming the FCFS (with commitment) allocation of components. Rather than enforcing a constraint that stipulates the service level of the ATO system, product-based backorder costs are introduced into the model. Assuming that the service parts are managed independently, the optimal base-stock levels for this multi-part-multi-product ATO system are derived. An approximation scheme is also proposed where product backordering cost rates are transformed into imputed item backordering cost rates. From a different perspective, Lu et al. (2010) study the kind of cost structures that renders no-hold-back allocation policies superior to all other allocation policies when items are managed independently using base-stock policies. A no-hold-back allocation policy is one that allows a job to be backordered if and only if at least one of its required parts is missing.

For periodic-review models, Gerchak et al. (1988) study the advantage of having common components for multiple products in a one-period ATO system. Such problem belongs to a class of general problems known as the newsvendor network problems (see e.g. Van Mieghem and Rudi (2002)). One-period ATO problems are two-stage stochastic programs with recourse where procurement decisions are made in the first stage followed by allocation in the second stage. Gerchak and Henig (2006) show that the one-period result extends to a myopic-allocation and base-stock-procurement policy which is optimal in the multiple-period case when the lead times are zero and sales are lost. Just as the simple newsvendor problem has its importance in studying a single-location-single-item periodic-review problem, the one-period ATO problem is important in studying a general ATO system. Similar to the continuous-review case, some work in the literature focuses on the estimation of performance measures while others also work on an optimization problem involving these performance measures. In terms of allocation, the order in which jobs arrive within the same period is not well defined and there exist variants of the FCFS policy. Base-stock levels are typically considered in the procurement of parts. In Hausman et al. (1998), the joint demand fill-rate is estimated and a constrained optimization problem involving a budget requirement is studied. In Zhang (2009), the total inventory in the system is minimized assuming that FCFS allocation and order-up-to procurement policies are employed and that minimal requirements on fill rates are enforced. A “fair-share” FCFS allocation policy is considered which “breaks the ties” among orders that arrived in the same period. Base stock levels that go with this allocation policy are optimized. Cheng et al. (2002) consider fill-rate constraints and an objective function that involves the holding cost of components.

The linear approximation and its resulting price-driven decomposition approach which we make use of in this chapter have their origins in other settings and are techniques used in approximate dynamic programming. Both Bertsekas and Tsitsiklis (1996) and Powell (2007) provide a great introduction and overview of the subject. The problem we deal with is weakly coupled according to Adelman and Mersereau (2008). In weakly coupled problems, the original problem can be decomposed by relaxing certain constraints that link the state of the system. An area of application where these techniques have been extensively employed is revenue management. For example, Topaloglu (2009), Kunnumkal and Topaloglu (2009) consider the use of such methods in network revenue management problems. In addition, the use of price-driven decomposition is considered in a job scheduling problem in Erdelyi and Topaloglu (2009) where jobs arrive at a workshop with finite daily capacities.

2.4 Dynamic Program Formulation

To capture the coupled demands for service parts, we define a set of recurring job types $\mathcal{J} = \{1, 2, ..., J\}$, a set of service parts $\mathcal{I} = \{1, 2, ..., I\}$ as well as a matrix R whose (i, j)-th entry, r_{ij}, corresponds to the number of parts of type i required by a job of type j. We call R the bill of materials. We let period 1 be the first period of the planning horizon and we let T be the last period in which a demand arises. Period T is also the last period in which orders for service parts can be placed. We allow the allocation of service parts to outstanding jobs up to period T' where $T' = T + \max_{i \in \mathcal{I}} L_i = T + \bar{L}$ and L_i is the deterministic order lead time of parts of type i. In other words, T is the procurement horizon and T' is the allocation horizon. We use f_j to denote the per-period backordering cost associated with a job of type j. In our discrete-time model, we assume that once a job is allocated all of its required service parts, it will incur no more charges in subsequent periods. We call the job repaired or completed. This is the same as assuming that assembly times are negligible in an assemble-to-order system.

Let h_i denote the per-period holding cost of service part i and let c_i denote the per-unit procurement cost of service part i. The total cost incurred during the entire length of the time horizon is simply the sum of all inventory purchasing and holding costs plus the sum of all penalty costs for delaying job completions.

The sequence of events in period t is as follows: (1) procured service parts arrive, (2) the random demand (arriving jobs) for period t, D^t (a J-component vector), is realized, and (3) allocation and procurement decisions are made for the current period. Following the arrival of procured parts, we use x^t (an I-component vector whose i-th entry is x_i^t) to denote the vector of all on-hand inventory and we use b^t (a J-component vector whose j-th entry is b_j^t) to denote the vector of all outstanding jobs. Note that these are the outstanding jobs which are carried over from previous time periods and do not include the demands to be realized in the current time period. Let D^t (with components D_j^t, $j = 1, ..., J$) denote the vector of realized demands. Furthermore, we use Q^t to capture the on-order inventory. In particular, Q^t is an I by $\bar{L}$ matrix whose (i, s)-th entry, q_i^{t+s}, is the number of parts of type i scheduled to arrive at the beginning of period $t + s$ (i.e. this order was placed at time $t + s - L_i$). We use q^{t+s} to denote the s-th column of Q. This is the vector of all service parts scheduled to arrive at time $t + s$.

To capture allocation decisions, we use a J-component vector, w^t. The j-th entry of w^t is denoted as w_j^t and it corresponds to the number of jobs of type j to repair in the current period. To capture procurement decisions, we use an I-

component vector $\tilde{q}^t$ whose i-th entry, $q_i^{t+L_i}$ corresponds to the number of service parts of type i to purchase in period t. Observe that we do not allocate parts to a job unless all the service parts necessary for job completion are allocated. Hence, the relevant decision is how many jobs to complete. Let $\mathcal{F}(x^t, b^t, D^t, R) = \{w \geq 0 : 0 \leq w \leq b^t + D^t, x^t - Rw \geq 0, w \text{ integers}\}$ be the space of all feasible allocation decisions given the current state of the system.

In terms of dynamics, the on-hand inventory after receiving the inventory scheduled to arrive at $t+1$ is given by:

$$x^{t+1} = x^t - Rw^t + q^{t+1} \tag{2.1}$$

and the number of backorders at $t+1$ before D^{t+1} is realized is given by

$$b^{t+1} = b^t + D^t - w^t. \tag{2.2}$$

As for Q^{t+1}, the matrix of on-order inventory, we start by defining e_l^k to be the k-component unit vector whose l-th component is 1. We set K_i to be $e_i^I (e_{L_i}^{\bar{L}})^T$ (an I by $\bar{L}$ matrix with a 1 in position (i, L_i) and 0 everywhere else). Now define S^L to be a shift-left operator for matrices that deletes the left most column of a matrix, shifts every entry to the left by one column, and appends a 0 column to the right. These allow us to write $Q^{t+1} = S^L(Q^t + \sum_{i \in \mathcal{I}} K_i q_i^{t+L_i})$.

The exact formulation of the **Equipment Overhaul Problem (EOP)** can be written as the following dynamic program recursion, for $1 \leq t \leq T'$:

$$\underline{\textbf{EOP:}} \quad V_t\left(x^t, b^t, D^t, Q^t\right) = \min_{(w^t, \tilde{q}^t) \in \mathcal{F}(x^t, b^t, D^t, R) \times (\mathbb{Z}_+ \cup \{0\})^I} \left\{ \sum_{j \in \mathcal{J}} f_j(b_j^t + D_j^t - w_j^t) \right.$$
$$+ \sum_{i \in \mathcal{I}} c_i q_i^{t+L_i} + \sum_{i \in \mathcal{I}} h_i \left(x_i^t - \sum_{j \in J} r_{ij} w_j^t \right)$$
$$\left. + \mathbb{E} V_{t+1} \left(x^t - Rw^t + q^{t+1}, b^t + D^t - w^t, D^{t+1}, S^L(Q^t + \sum_{i \in \mathcal{I}} K_i q_i^{t+L_i}) \right) \right\} \tag{2.3}$$

where $V_{T'+1} \equiv 0$, and for $t > T$, we set D^t to be deterministically zero and enforce $\tilde{q}^t$ to be equal to zero.

The sizes of the state space and decision space render the exact formulation difficult to solve in a computationally tractable manner. We discuss approximate solutions in the rest of this chapter.

2.5 Deterministic and Randomized Linear Programs

Because we have assumed linear holding, acquisition, and backorder costs, the equipment overhaul problem could be formulated as a linear program if the demands were known quantities at the beginning of the time horizon and if we relax the integer constraints. Using the same notation as defined in the previous section, the deterministic equipment overhaul problem for periods t through T' is given below, with further explanations to follow:

$$\underline{\textbf{Deterministic EOP:}} \quad \min \sum_{j \in \mathcal{J}} \sum_{t \leq \tau \leq T'} f_j \, b_j^{\tau+1} + \sum_{i \in \mathcal{I}} \sum_{t \leq \tau \leq T'} \left(h_i \tilde{x}_i^{\tau} + c_i q_i^{\tau+L_i} \right), \quad (2.4)$$

$$\text{subject to} \quad \sum_{j\in\mathcal{J}} r_{ij} w_j^t + \tilde{x}_i^t = x_i^t, \quad i \in \mathcal{I}, \tag{2.5}$$

$$\sum_{j\in\mathcal{J}} r_{ij} w_j^\tau - \tilde{x}_i^{\tau-1} + \tilde{x}_i^\tau = q_i^\tau, \quad t < \tau < t + L_i,\ i \in \mathcal{I}, \tag{2.6}$$

$$\sum_{j\in\mathcal{J}} r_{ij} w_j^\tau - \tilde{x}_i^{\tau-1} - q_i^\tau + \tilde{x}_i^\tau = 0, \quad t + L_i \le \tau \le T + L_i,\ i \in \mathcal{I}, \tag{2.7}$$

$$\sum_{j\in\mathcal{J}} r_{ij} w_j^\tau - \tilde{x}_i^{\tau-1} + \tilde{x}_i^\tau = 0, \quad T + L_i < \tau \le T',\ i \in \mathcal{I}, \tag{2.8}$$

$$w_j^t + b_j^{t+1} = b_j^t + D_j^t, \quad j \in \mathcal{J}, \tag{2.9}$$

$$w_j^\tau - b_j^\tau + b_j^{\tau+1} = D_j^\tau, \quad t + 1 \le \tau \le T',\ j \in \mathcal{J}, \tag{2.10}$$

$$q_i^\tau \ge 0,\ i \in \mathcal{I},\ t + L_i \le \tau \le T + L_i, \quad \tilde{x}_i^\tau \ge 0,\ i \in \mathcal{I},\ t \le \tau \le T',$$
$$w_j^\tau \ge 0,\ j \in \mathcal{J},\ t \le \tau \le T', \quad b_j^\tau \ge 0,\ j \in \mathcal{J},\ t < \tau \le T' + 1.$$

The objective function corresponds to the total cost incurred from period t until period T'. Holding costs and backorder costs are charged based on the on-hand inventory and outstanding backorders at the end of each time period. With x_i^τ defined to be the amount of on-hand inventory of part i at the beginning of period τ, $\tilde{x}_i^\tau$ corresponds to the amount of on-hand inventory of part i at the end of period τ after allocation decisions are made. These appear in the above optimization problem as decision variables because they depend on the allocation and procurement decisions. The decision variable $b_j^{\tau+1}$ corresponds to the number of backorders of type j left at the beginning of period $\tau + 1$ and it is dependent on the allocation and procurement decisions made in previous time periods. Because b_j^τ is defined to the the number of outstanding backorders of type j at the beginning of time period τ before demand D^t is realized, b_j^τ also corresponds to the number of outstanding backorders at the end of $\tau - 1$ after allocation decisions are made.

Terms consisting of decision variables (allocation and procurement decisions, as well as on-hand inventory and backorders after receiving inventory and allocation) are included on the left-hand side of the constraint equations and parameters related to the state of the system at the beginning of period t (on-hand inventory, on-order inventory, existing backorders and the demand just realized) are included on the right-hand side. In particular, note that q_i^τ is a parameter if $t < \tau < t + L_i$ since it is part of Q^t. Otherwise, if $t + L_i \leq \tau \leq T + L_i$, it is a decision variable as it pertains to a procurement decision to be made now or later in the time horizon.

Equations (2.5) to (2.8) are balance equations pertaining to the service parts in the system from time t to T'. The ending inventory in period τ is equal to the ending inventory in period $\tau - 1$ plus the inventory arriving at the beginning of τ minus the inventory allocated to outstanding jobs in period τ. Similarly, equations (2.9) to (2.10) are balance equations pertaining to the backorders in the system from period t through period T'. The number of backorders at the beginning of $\tau + 1$ is equal to the number of backorders at the beginning of τ plus the demand realized at τ minus the number of jobs completed in period τ.

Using a result from the stochastic programming literature (Birge and Louveaux, 1997), an immediate lower-bound result is given below:

Proposition 2.1. *Using $L(x^t, b^t, D^t, Q^t, \{D^\tau\}_{t<\tau\leq T'})$ to denote the optimal objective value of the deterministic EOP, we have*

$$\begin{aligned} L(x^t, b^t, D^t, Q^t, \{\mathbb{E}D_j^\tau\}_{j\in\mathcal{J},\ t<\tau\leq T'}) &\leq \mathbb{E}L(x^t, b^t, D^t, Q^t, \{D_j^\tau\}_{j\in\mathcal{J}\ t<\tau\leq T'}) \\ &\leq V_t(x^t, b^t, D^t, Q^t). \end{aligned}$$

Suppose we solve the deterministic EOP with demands replaced by their expected values and obtain dual variables $-\mu_i^\tau$ for the first three sets of constraints (2.5) to (2.8) and λ_j^τ for the remaining sets (2.9) to (2.10). Note that μ_i^τ has the interpretation of the value of having a service part of type i in the system at time τ, while λ_j^τ has the interpretation of the cost of having a job of type j backordered at time τ. By the proposition above, we may linearly approximate from below the value function for the current period t using:

$$\begin{aligned} V_t(x^t, b^t, D^t, Q^t) &\approx \sum_{i\in\mathcal{I}} \left\{ -\mu_i^t x_i^t - \sum_{\tau=t+1}^{t+L_i-1} \mu_i^\tau q_i^\tau \right\} \\ &\quad + \sum_{j\in\mathcal{J}} \left\{ \lambda_j^t (b_j^t + D_j^t) + \sum_{\tau=t+1}^{T'} \lambda_j^\tau \mathbb{E}[D_j^\tau] \right\} \\ &= L(x^t, b^t, D^t, Q^t, \{\mathbb{E}D_j^\tau\}_{j\in\mathcal{J},\, t<\tau\le T'}) \end{aligned}$$

Given this linear approximation, we may approximate $\mathbb{E}V_{t+1}$ appearing in (2.3) using:

$$\begin{aligned} \mathbb{E}V_{t+1}(x^{t+1}, b^{t+1}, D^{t+1}, Q^{t+1}) \approx & \sum_{i\in\mathcal{I}} \left\{ -\mu_i^{t+1} x_i^{t+1} - \sum_{\tau=t+2}^{t+L_i} \mu_i^\tau q_i^\tau \right\} \\ & + \sum_{j\in\mathcal{J}} \left\{ \lambda_j^{t+1} (b_j^{t+1} + \mathbb{E}D_j^{t+1}) + \sum_{\tau=t+2}^{T'} \lambda_j^\tau \mathbb{E}[D_j^\tau] \right\} \end{aligned}$$

Noting that the on-hand inventory and the number of backorders evolve according to equations (2.1) and (2.2) which are both functions of the current state and decision variables, we may substitute them into the above approximation and then plug the result into the Bellman's recursion (neglecting all constant terms not involving decision variables) to arrive at the following single period integer program which can be used to drive our decisions:

Multi-Job, Multi-Part, Single-Period Approximation Problem:

$$\min \quad \sum_{j\in\mathcal{J}} \left(-f_j - \sum_{i\in\mathcal{I}} h_i r_{ij} + \sum_{i\in\mathcal{I}} \mu_i^{t+1} r_{ij} - \lambda_j^{t+1} \right) w_j^t + \sum_{i\in\mathcal{I}} \left(c_i - \mu_i^{t+L_i} \right) q_i^{t+L_i} \tag{2.11}$$

$$
\begin{aligned}
\text{subject to} \quad & 0 \le w_j^t \le b_j^t + D_j^t, \ \forall j \in \mathcal{J}, \\
& q_i^{t+L_i} \ge 0, \ \forall i \in \mathcal{I}, \\
& \sum_{j \in J} r_{ij} w_j^t \le x_i^t, \ \forall i \in \mathcal{I}, \qquad (2.12) \\
& q_i^{t+L_i}, w_j^t \text{ integers}, \ \forall i \in \mathcal{I}, \ \forall j \in \mathcal{J}.
\end{aligned}
$$

The solution to this integer program provides us with a guide on how to allocate the on-hand inventory among outstanding backorders. Because we have used the dual variables from the deterministic linear program to drive our allocation decisions, we call it a price-driven allocation policy. It is our conjecture that such an allocation policy will outperform a myopic allocation policy and other standard allocation policies assumed in the literature such as the first-come-first-served policy.

Looking at the integer program obtained above, one quickly notices that the linearization of value functions renders procurement decisions indeterminate. According to this first-order approximation, the policy would choose to purchase nothing if $c_i - \mu_i^{t+L_i} > 0$. With the dual constraints

$$
c_i - \mu_i^{t+L_i} \ge 0, \qquad (2.13)
$$

it is not possible to tell how much one should order when $c_i - \mu_i^{t+L_i} = 0$ (i.e. when c_i is equal to the value of a part of type i a lead time from now). To solve this problem, we consider a tighter but nonlinear approximation of the value function in the next section.

In view of the result stated in Proposition 2.1, instead of solving the linear program just once with the demand parameters replaced by their expected values, we may solve randomized linear programs by sampling future demands and use the average of the dual values obtained instead. Because the expected value of the randomized LP optimum provides a tighter bound for the value function, we will make use of randomized linear programs when we use the price-driven allocation policy.

We conclude this section by restating the optimization problem we solve in each period to obtain the allocation decisions under our proposed policy:

Multi-Job, Multi-Part, Allocation Problem:

$$\min \quad \sum_{j\in\mathcal{J}} \left(-f_j - \sum_{i\in\mathcal{I}} h_i r_{ij} + \sum_{i\in\mathcal{I}} \overline{\mu}_i^{t+1} r_{ij} - \overline{\lambda}_j^{t+1} \right) w_j^t \tag{2.14}$$

$$\begin{aligned} \text{subject to} \quad & 0 \le w_j^t \le b_j^t + D_j^t,\ \forall j \in \mathcal{J}, \\ & \sum_{j\in J} r_{ij} w_j^t \le x_i^t,\ \forall i \in \mathcal{I}, \\ & w_j^t \text{ integers},\ \forall j \in \mathcal{J}. \end{aligned} \tag{2.15}$$

Here, $\overline{\mu}_i^{t+1}$ and $\overline{\lambda}_j^{t+1}$ correspond to the averages of the dual values obtained after solving many instances of the deterministic EOP with sampled future demands. Observe that a myopic allocation policy would correspond to one that gives priority to the job type with the highest per-period backorder cost, f_j, plus the per-period holding cost of all of its required service parts, $\sum_{i\in\mathcal{I}} h_i r_{ij}$. The incorporation of $\overline{\lambda}_j^{t+1}$ helps augment this myopic cost by the cost of having this job type backordered at the end of the next time period. Furthermore, the incorporation of $\sum_{i\in\mathcal{I}} \overline{\mu}_i^{t+1} r_{ij}$ helps discount this myopic cost by the value of having certain service parts available in the next time period, should we not repair this job type in the current period.

2.6 Decomposition by Service Part Type

It is possible to get a tighter bound than that given by the deterministic linear program in (2.4) where the parameters D_j^τ are replaced by their expected values. We let $\{-\mu_i^\tau\}_{i\in\mathcal{I},t\le\tau\le T'}$ be the dual variables associated with the balance equations for the various service parts in this deterministic linear program. We dualize all of these constraints except for those associated with a specific service part $l \in \mathcal{I}$. Doing this, we get:

$$\begin{aligned}
\min \quad & \sum_{j\in\mathcal{J}} \sum_{t\le\tau\le T'} \left(f_j + \sum_{i\in\mathcal{I}\setminus\{l\}} r_{ij} \left(\mu_i^{\tau+1} \mathbb{1}_{\{\tau<T'\}} - \mu_i^\tau \right) \right) b_j^{\tau+1} \\
& + \sum_{t\le\tau\le T'} \left(h_l \tilde{x}_l^\tau + c_l q_l^{\tau+L_l} \right) + \sum_{i\in\mathcal{I}\setminus\{l\}} \sum_{t\le\tau\le T'} \left(h_i - \mu_i^{\tau+1} \mathbb{1}_{\{\tau<T'\}} + \mu_i^\tau \right) \tilde{x}_i^\tau \\
& + \sum_{i\in\mathcal{I}\setminus\{l\}} \sum_{t\le\tau\le T} \left(c_i - \mu_i^{\tau+L_i} \right) q_i^{\tau+L_i} + \sum_{i\in\mathcal{I}\setminus\{l\}} \left\{ (-\mu_i^t) x_i^t - \sum_{t<\tau<t+L_i} \mu_i^\tau q_i^\tau \right. \\
& \left. + \mu_i^t \sum_{j\in\mathcal{J}} r_{ij} \left(b_j^t + D_j^t \right) + \sum_{t<\tau\le T'} \mu_i^\tau \sum_{j\in\mathcal{J}} r_{ij} \mathbb{E} D_j^\tau \right\}.
\end{aligned}$$

The remaining constraints for this relaxed linear program are analogous to those found in (2.5)- (2.10). The only difference is that we are now focused on only one of the I service parts:

$$\sum_{j\in\mathcal{J}} r_{lj}w_j^t + \tilde{x}_l^t = x_l^t,$$

$$\sum_{j\in\mathcal{J}} r_{lj}w_j^\tau - \tilde{x}_l^{\tau-1} + \tilde{x}_l^\tau = q_l^\tau, \quad t<\tau<t+L_l,$$

$$\sum_{j\in\mathcal{J}} r_{lj}w_j^\tau - \tilde{x}_l^{\tau-1} - q_l^\tau + \tilde{x}_l^\tau = 0, \quad t+L_l \le \tau \le T+L_l,$$

$$\sum_{j\in\mathcal{J}} r_{lj}w_j^\tau - \tilde{x}_l^{\tau-1} + \tilde{x}_l^\tau = 0, \quad T+L_l<\tau\le T',$$

$$w_j^t + b_j^{t+1} = b_j^t + D_j^t, \quad j\in\mathcal{J},$$

$$w_j^\tau - b_j^\tau + b_j^{\tau+1} = D_j^\tau, \quad t+1\le\tau\le T',\ j\in\mathcal{J},$$

$$q_i^\tau \ge 0,\ i\in\mathcal{I},\ t+L_i\le\tau\le T+L_i,\ \ \tilde{x}_i^\tau\ge 0,\ i\in\mathcal{I},\ t\le\tau\le T',$$
$$w_j^\tau\ge 0,\ j\in\mathcal{J},\ t\le\tau\le T',\ \ b_j^\tau\ge 0,\ j\in\mathcal{J},\ t<\tau\le T'+1.$$

Note that because all feasible solutions to the non-relaxed LP are still feasible for this relaxed problem with identical objective values, the solution to the relaxed problem gives an overall lower bound because of an enlarged feasible region. It is also worth pointing out that expression enclosed in curly brackets which appears last in the objective function consists of constant terms which are independent of the decision variables. These terms are linear in the state variables $\{x_i^t\}_{i\in\mathcal{I}\setminus\{l\}}$, b^t and D^t as well as the rows of Q^t for $i\in\mathcal{I}\setminus\{l\}$.

It turns out that

$$\left(h_i - \mu_i^{\tau+1}\mathbb{1}_{\{\tau<T'\}} + \mu_i^\tau\right) \ge 0 \tag{2.16}$$

and $\left(c_i - \mu_i^{\tau+L_i}\right) \ge 0$, as given in (2.13), in the objective function by the dual constraints of the deterministic EOP. Associated with these coefficients are the decision variables $\tilde{x}_i^\tau$ and $q_i^{\tau+L_i}$ for $i\in\mathcal{I}\setminus\{l\}$. Because these non-negative decision variables do not appear anywhere in the constraints and this is a minimization problem, they must all be set to zero in the optimal solution. Ignoring these terms in the objective function, we now have a linear program which is identical in form to the original non-relaxed linear program, except that it is for a single service part and the objective function coefficients have been modified. We now have only the balance equations for a specific service part in the constraints. Using Proposition 2.1 in the previous section, we recognize that this relaxed linear program above provides a lower bound for a value function satisfying the following Bellman's recursion:

Single-Part, Multi-Job-Type Approximate DP:

$$\begin{aligned} & v_t^l\left(x_l^t, b^t, D^t, Q_l^t\right) \\ &= \min_{\substack{(w^t, q_l^{t+L_l}) \in \\ \mathcal{F}(x_l^t, b^t, D^t, R_l) \times (\mathbb{Z}_+ \cup \{0\})}} \left\{ \sum_{j \in \mathcal{J}} \left(f_j + \sum_{i \in \mathcal{I} \setminus \{l\}} r_{ij} \left(\mathbb{1}_{\{t<T'\}} \mu_i^{t+1} - \mu_i^t \right) \right) \left(b_j^t + D_j^t - w_j^t \right) \right. \\ & \left. + c_l q_l^{t+L_l} + h_l \left(x_l^t - R_l w^t \right) + \mathbb{E} v_{t+1}^l \left(x_l^t - R_l w^t + q_l^{t+1}, b^{t+1}, D^{t+1}, Q_l^{t+1} \right) \right\}. \end{aligned} \tag{2.17}$$

Here, R_l and Q_l^t are row vectors extracted from R and Q^t. Note that the dual values μ_i^t are obtained by solving the deterministic linear program in (2.4) once by replacing the random variables D^t with their expected values. We call Formulation (2.17) the **single-part-multi-job-type problem**. With the single-part-multi-job-type problem defined, we give a second lower bound result below.

Proposition 2.2. *Decomposing $V_t(x^t, b^t, D^t, Q^t)$ into single-part-multi-job-type problems gives rise to a tighter lower bound than the value obtained using the deterministic* *linear program. In particular,*

$$
\begin{aligned}
& L(x^t, b^t, D^t, Q^t, \{\mathbb{E}D_j^\tau\}_{j\in\mathcal{J},\ t<\tau\le T'}) \\
\le\ & v_t^l\left(x_l^t, b^t, D^t, Q_l^t\right) + \sum_{i\in\mathcal{I}\setminus\{l\}} \Bigg\{ (-\mu_i^t)x_i^t - \sum_{t<\tau<t+L_i} \mu_i^\tau q_i^\tau \\
& \quad + \mu_i^t \sum_{j\in\mathcal{J}} r_{ij}\left(b_j^t + D_j^t\right) + \sum_{t<\tau\le T'} \mu_i^\tau \sum_{j\in\mathcal{J}} r_{ij}\mathbb{E}D_j^\tau \Bigg\} \\
\le\ & V_t(x^t, b^t, D^t, Q^t).
\end{aligned}
$$

Proof: The first inequality follows from Proposition 2.1 as explained. The second inequality can be shown using induction. We define the feasible set $\mathcal{G}(x^t, b^t, D^t, R) = \{(w^t, \tilde{x}^t) \ge 0 : Rw_j^t + \tilde{x}^t = x^t, w^t \le b^t + D^t\}$. Assume as the induction hypothesis the result for $t+1$. Recall that we have V_t defined as:

$$
\begin{aligned}
& V_t\left(x^t, b^t, D^t, Q^t\right) \\
=\ & \min_{\substack{(w^t,\tilde{x}^t,\tilde{q}^t)\in \\ \mathcal{G}(x^t,b^t,D^t,R)\times(\mathbb{Z}_+\cup\{0\})^I}} \Bigg\{ \sum_{j\in\mathcal{J}} f_j(b_j^t + D_j^t - w_j^t) + \sum_{i\in\mathcal{I}} c_i q_i^{t+L_i} + \sum_{i\in\mathcal{I}} h_i \tilde{x}_i^t \\
& \quad + \mathbb{E}V_{t+1}\left(x^{t+1}, b^{t+1}, D^{t+1}, Q^{t+1}\right) \Bigg\}.
\end{aligned}
$$

In addition to using the induction hypothesis, we first relax all the part-type constraints associated with $i \in \mathcal{I} \setminus \{l\}$. The constraint for part $i \in \mathcal{I} \setminus \{l\}$ is dualized with $-\mu_i^t$ and incorporated into the objective function. Doing this, we get that

$$
\begin{aligned}
&V_t\left(x^t, b^t, D^t, Q^t\right) \\
&\geq \min_{\substack{(w^t,\tilde{x}_l^t,\tilde{q}^t)\in \\ \mathcal{G}(x_l^t,b^t,D^t,R_l)\times(\mathbb{Z}_+\cup\{0\})^I}} \left\{\sum_{j\in\mathcal{J}} f_j(b_j^t + D_j^t - w_j^t) + \sum_{i\in\mathcal{I}} c_i q_i^{t+L_i} + \sum_{i\in\mathcal{I}} h_i \tilde{x}_i^t \right. \\
&\quad +\mathbb{E}\left[v_{t+1}^l(x_l^{t+1}, D^{t+1}, B^{t+1}, Q_l^{t+1}) + \sum_{i\in\mathcal{I}\setminus\{l\}} \left\{ (-\mu_i^{t+1})x_i^{t+1} - \sum_{\tau=t+2}^{t+L_i} \mu_i^\tau q_i^\tau \right.\right. \\
&\quad \left.\left. +\mu_i^{t+1}\sum_{j\in\mathcal{J}} r_{ij}(D_j^{t+1} + b_j^{t+1}) + \sum_{\tau=t+2}^{T'} \mu_i^\tau \sum_{j\in\mathcal{J}} r_{ij}\mathbb{E}D_j^\tau \right]\right\} \\
&\quad \left. - \sum_{i\in\mathcal{I}\setminus\{l\}} \mu_i^t \left(x_i^t - \sum_{j\in\mathcal{J}} r_{ij} w_j^t - \tilde{x}_i^t \right) \right\}.
\end{aligned}
$$

Note again that the last term appears here because of the dualization of some of the constraints. We now substitute in the evolution equations for x^{t+1} and b^{t+1} using (2.1) and (2.2). Furthermore, we add and subtract $\sum_{i\in\mathcal{I}\setminus\{l\}} \mu_i^t \sum_{j\in\mathcal{J}} r_{ij}(b_j^t + D_j^t)$. A careful rearrangement of the terms will yield that the right hand side of the above is equivalent to the right hand side of the following:

$$
\begin{aligned}
&V_t\left(x^t, b^t, D^t, Q^t\right) \\
&\geq \sum_{i\in\mathcal{I}\setminus\{l\}} \left\{ -\mu_i^t x_i^t - \sum_{t<\tau<t+L_i} \mu_i^\tau q_i^\tau + \mu_i^t \sum_{j\in\mathcal{J}} r_{ij}(b_j^t + D_j^t) + \sum_{\tau=t+1}^{T'} \mu_i^\tau \sum_{j\in\mathcal{J}} r_{ij}\mathbb{E}D_j^\tau \right\} \\
&\quad + \min_{\substack{(w^t,\tilde{x}_l^t,\tilde{q}^t)\in \\ \mathcal{G}(x_l^t,b^t,D^t,R_l) \\ \times(\mathbb{Z}_+\cup\{0\})^I}} \left\{ \sum_{j\in\mathcal{J}} \left(f_j + \sum_{i\in\mathcal{I}\setminus\{l\}} r_{ij}\left(\mathbb{1}_{\{t<T'\}}\mu_i^{t+1} - \mu_i^t\right) \right) (b_j^t + D_j^t - w_j^t) \right. \\
&\quad + c_l q_l^{t+L_l} + h_l \tilde{x}_l^t + \mathbb{E}v_{t+1}^l\left(x_l^{t+1}, b^{t+1}, D^{t+1}, Q_{l\cdot}^{t+1}\right) + \sum_{i\in\mathcal{I}\setminus\{l\}} (c_i - \mu_i^{t+L_i}) q_i^{t+L_i} \\
&\quad \left. + \sum_{i\in\mathcal{I}\setminus\{l\}} \left(h_i - \mu_i^{t+1}\mathbb{1}_{\{t<T'\}} + \mu_i^t\right)\tilde{x}_i^t \right\}.
\end{aligned}
$$

Note that as given by the dual constraints (2.13) and (2.16) of the deterministic EOP, the coefficients of $\tilde{x}_i^t$ and $q_i^{t+L_i}$ are non-negative. Because $\tilde{x}_i^t$ and $q_i^{t+L_i}$ for $i \in \mathcal{I}\setminus\{l\}$ do not interact with any other constraints in the minimization problem, they must be equal to zero in the optimal solution. The right hand side of the above can therefore be shown to be equal to the right hand side of the following:

$$
\begin{aligned}
& V_t\left(x^t, b^t, D^t, Q^t\right) \\
\geq \quad & \sum_{i \in \mathcal{I} \backslash \{l\}} \left\{ (-\mu_i^t) x_i^t - \sum_{t < \tau < t + L_i} \mu_i^\tau q_i^\tau + \mu_i^t \sum_{j \in \mathcal{J}} r_{ij} \left(D_j^t + b_j^t\right) \right. \\
& \left. + \sum_{t < \tau \leq T'} \mu_i^\tau \sum_{j \in \mathcal{J}} r_{ij} \mathbb{E} D_j^\tau \right\} + v_t^l \left(x_l^t, b^t, D^t, Q_l^t\right).
\end{aligned}
$$

This completes our proof. □

This result shows that we can approximate the original value function with a value function which depends only on the vector of outstanding jobs and the inventory of one service part plus linear terms associated with other service parts. The value function corresponds to the single-part-multi-job-type problem defined in (2.17) and the linear terms are dependent on the state variables for all the other service parts. Although the state space dimension of the new dynamic program is much reduced compared to the original problem, it is still too difficult to solve exactly using backward recursion for large problem sizes. In the next section, we discuss a second layer of decomposition that allows us to solve this problem in a computationally tractable manner.

2.7 The Single-Part-Multi-Job-Type Problem

In this section, we focus on the single-part-multi-job-type problem (2.17) that arises from the approximation of the original value function described in the previous section:

$$
\begin{aligned}
&v_t^l\left(x_l^t, b^t, D^t, Q_l^t\right)\\
&= \min_{\substack{(w^t, q_l^{t+L_l}) \in \\ \mathcal{F}(x_l^t, b^t, D^t, R_l)\times(\mathbb{Z}_+\cup\{0\})}} \left\{ \sum_{j\in\mathcal{J}} \left(f_j + \sum_{i\in\mathcal{I}\setminus\{l\}} r_{ij}\left(\mathbb{1}_{\{t<T'\}}\mu_i^{t+1} - \mu_i^t\right)\right)\left(b_j^t + D_j^t - w_j^t\right)\right.\\
&\qquad \left. + c_l q_l^{t+L_l} + h_l\left(x_l^t - R_l w^t\right) + \mathbb{E} v_{t+1}^l\left(x_l^t - R_l w^t + q_l^{t+1}, b^{t+1}, D^{t+1}, Q_l^{t+1}\right)\right\}.
\end{aligned}
$$

Note that with the dual variables included, the per-period backordering cost becomes dependent on time. Depending on the magnitude of the dual variables, these backorder costs may become negative. When this cost is negative, it is valuable to backorder the job for an extra period. This could translate to reserving inventory for a job type with higher-priority demands anticipated to arrive in the near future. Alternately, this could mean that other service parts needed for job completion in the original problem are not available until later. In this section, we describe how to use the Clark-and-Scarf decomposition approach (Clark and Scarf, 1960) to solve the single-part-multi-job-type problem. The Clark-and-Scarf decomposition was developed in the context of a multi-echelon inventory system. The single-part-multi-job-type problem is analogous to a two-echelon distribution problem consisting of a central order location and a number of demand nodes where demands arise and where backorders accumulate. The lead time at the central order location corresponds to the order lead time of the service part but unlike the analogous distribution system, the demand nodes have no order lead times because "shipments" to these demand nodes from the central order location correspond to the (instantaneous) allocation of service parts among different job types.

We need to make one further relaxation in order to apply the Clark-and-Scarf decomposition. As presented above and in (2.17), the single-part-multi-job-type problem distinguishes between the service part and the various job types. In this section, we assume that outstanding jobs are kept track of as outstanding demands for service parts. Since r_{lj} corresponds to the number of parts of type l required by a job of type j, if there exist $b_j^t + D_j^t$ backorders of type j, there are $r_{lj}(b_j^t + D_j^t)$ outstanding orders for parts of type l. Similarly, repairing w_j^t jobs of type j means using up $r_{lj}w_j^t$ parts of type l. Replacing w_j^t with $\tilde{w}_j^t = r_{lj}w_j^t$, and also b_j^t and D_j^t with $\tilde{b}_j^t = r_{lj}b_j^t$ and $\tilde{D}_t^j = r_{lj}D_t^j$ respectively, we obtain the following modified single-part-multi-job-type problem:

$$
\begin{aligned}
&v_t^l\left(x_l^t, \tilde{b}^t, \tilde{D}^t, Q_l^t\right) \\
&\quad = \min_{\substack{\tilde{w}^t \text{ integer}\geq 0,\\ 0\leq \tilde{w}^t \leq \tilde{b}^t + \tilde{D}^t,\\ x^t - \tilde{w}^t \geq 0,\\ q_l^{t+L_l}\geq 0.}} \left\{ \sum_{j\in\mathcal{J}} \frac{f_j + \sum_{i\in\mathcal{I}\setminus\{l\}} r_{ij}\left(\mathbb{1}_{\{t<T'\}}\mu_i^{t+1} - \mu_i^t\right)}{r_{lj}} (\tilde{b}_j^t + \tilde{D}_j^t - \tilde{w}_j^t) \right. \\
&\qquad \left. + c_l q_l^{t+L_l} + h_l\left(x_l^t - \tilde{w}^t\right) + \mathbb{E}v_{t+1}^l\left(x_l^t - \tilde{w}^t + q_l^{t+1}, \tilde{b}^{t+1}, \tilde{D}^{t+1}, Q_l^{t+1}\right) \right\}. \qquad (2.18)
\end{aligned}
$$

We assume without loss of generality here that $r_{lj} > 0$ for all $j \in \mathcal{J}$. If this is not the case, jobs with $r_{lj} = 0$ can simply be removed from the set $\mathcal{J}$. The modified single-part-multi-job-type problem differs in that both x_l^t and $\tilde{b}^t$ are counted in terms of service parts. While it is possible to characterize the distribution of $\tilde{D}_t^j = r_{lj}D_t^j$ exactly, the modified single-part-multi-job-type problem potentially admits solutions which consist of the allocation of only some of the r_{lj} service parts of type l required by a job of type j. It is therefore a relaxation of the original single-part-multi-job type problem.

We consider the modified single-part-multi-job-type subproblem consisting only of the generic service part $l \in \mathcal{I}$ in the rest of the section. Because there is only one service part involved, we **suppress the subscript** $l \in \mathcal{I}$ for clarity in this section. There remain multiple job types in this subproblem and we use subscript $j \in \mathcal{J}$ to distinguish among them. We continue to let T be the last period in which a demand arises and we continue to allow the allocation of parts to outstanding jobs until $T' := T + \bar{L}$ where $\bar{L}$ is a number greater than or equal to the order lead time, L, of the service part. (In decomposing the original problem into many single-part-multi-job-type subproblems, we set $\bar{L}$ to be $\max_{i \in \mathcal{I}}\{L_i\}$.)

We start by introducing some notation that will help us more succinctly write out the modified single-part-multi-job-type problem. We let z_0^t be the net inventory of the generic service part after the demand in period t is realized.

Furthermore, we let $z_j^t, j = 1, ..., J$ be the net inventory level of the generic service part committed to jobs of type j in period t after the demand in this period has been realized. Much like the notion of echelon net inventory in the context of multi-echelon systems, the advantage of having z_0^t is that it evolves in a manner independent of the allocation decisions. While $\tilde{w}_j^t$ decreases the number of backorders of type j, it also decreases the on-hand inventory. Since net-inventory is defined to be on-hand inventory minus backorders, the allocation decision $\tilde{w}_j^t$ does not affect the dynamics of z_0^t.

Observe that $(z_j^t)^-$, the negative part of z_j^t, captures the number of backorders of type j that are outstanding prior to allocation in period t. Therefore, z_j^t as defined is equal to $-(\tilde{b}_t^j + \tilde{D}_j^t) = -r_{lj}(b_j^t + D_j^t)$ if there are no service parts committed to jobs of type j. Note that $z_0^t - \sum_{j \in \mathcal{J}} z_j^t$ represents the number of units that are on-hand which are not already committed to a job type after the demand in period t is realized.

We let $Q^t = (q^{t+1}, q^{t+2}, ..., q^{t+L-1})$ be the vector of on-order units for the generic service part at time t. The quantity q^s was ordered at $s - L$ and it is scheduled to arrive at time s. The sequence of events at t (where $1 \leq t \leq T'$) is as follows:

1. The order placed a lead time ago at $t - L$ arrives.
2. The demand vector $\tilde{D}^t = (\tilde{D}^t_0, \tilde{D}^t_1, ..., \tilde{D}^t_J)$ where $\tilde{D}^t_0 = \sum_{j=1}^{J} \tilde{D}^t_j$ is observed. (Note that the random variables $\tilde{D}^t_j, j = 1, ..., J$ could be correlated. Also, note that $\tilde{D}^t_j = 0$ for $j \in \mathcal{J}$ and for $T < t \leq T'$.)
3. The vector $z^t = (z^t_0, z^t_1, ..., z^t_J)$ is updated. This vector keeps track of the net-inventory of the system as well as the pre-allocation backorders of each type.
4. The vector $\tilde{w}^t = (\tilde{w}^t_1, \tilde{w}^t_2, ..., \tilde{w}^t_J)$ captures the allocation decisions. We also decide to procure $q^{t+L} \geq 0$ if $t \leq T$. Note that with $z^t_0 - \sum_{j \in \mathcal{J}} z^t_j$ representing the number of units that are on-hand which are not already committed to a job type after the demand in period t is realized, we need to ensure that $\sum_{j \in \mathcal{J}} \tilde{w}^t_j \leq z^t_0 - \sum_{j \in \mathcal{J}} z^t_j$.

If we start with $z^t_j \leq 0$ in period t after demand is realized, we can restrict the decision variable $\tilde{w}^t_j$ in such a way that $z^t_j + \tilde{w}^t_j \leq 0$. Setting $w^t_j > (z^t_j)^-$ means committing a service part to jobs of type j before the jobs actually materialize. This cannot help to lower cost. The one-period backorder cost of type j incurred after allocation is, therefore, equal to $\tilde{f}^t_j((z^t_j + \tilde{w}^t_j)^-) = -\tilde{f}^t_j(z^t_j + \tilde{w}^t_j)$ at time t where $\tilde{f}^t_j$ is equal to $\frac{f_j + \sum_{i \in \mathcal{I} \setminus \{l\}} r_{ij} \left(\mathbb{1}_{\{t < T'\}} \mu^{t+1}_i - \mu^t_i \right)}{r_{lj}}$ which appears in Formulation (2.18). Note once again the dependence of $\tilde{f}^t_j$ on t. Here, $(z^t_j + \tilde{w}^t_j)^-$ is the number of backorders of type j which are left in period t after allocation decisions are made. At the end of period t, on-hand inventory is equal to net-inventory plus backorders which is $z_0^t - \sum_{j\in\mathcal{J}}(z_j^t + \tilde{w}_j^t)$. Therefore, the one-period holding cost for period t is equal to $h\left[z_0^t - \sum\limits_{j\in\mathcal{J}}(z_j^t + \tilde{w}_j^t)\right]$ where h is the per-period per-unit holding cost. We now define

$$L_{jt}(z_j^t + \tilde{w}_j^t) = (-\tilde{f}_j^t - h)(z_j^t + \tilde{w}_j^t), \quad z_j^t + \tilde{w}_j^t \leq 0,\ j = 1, ..., J,\ t = 1, ..., T', \tag{2.19}$$

and

$$L_{0t}(z_0^t) = h(z_0^t), \quad t = 1, ..., T'. \tag{2.20}$$

We let $y_j^t = z_j^t + \tilde{w}_j^t$ be the allocation level for $j = 1, ..., J$. With c defined to be the per unit procurement cost, the dynamic program for the modified single-part-multi-job-type problem in (2.18) can be rewritten as

$$\begin{aligned} V_t\left(z^t, Q^t\right) = \min_{\substack{z_j^t \leq y_j^t \leq 0,\ j\in\mathcal{J}, \\ q^{t+L}\geq 0 \text{ if } t\leq T,\ q^{t+L}=0 \text{ if } t>T, \\ \sum\limits_{j\in\mathcal{J}} y_j^t \leq z_0^t.}} & \left\{ cq^{t+L} + L_{0t}(z_0^t) + \sum_{j=1}^{J} L_{jt}(y_j^t) \right. \\ & \left. + \mathbb{E}V_{t+1}\left(z^{t+1}(z^t, y^t, q^{t+1}, \tilde{D}^t), Q^{t+1}\right) \right\} \end{aligned} \tag{2.21}$$

with boundary condition $V_{T'+1} = 0$. The constraint $\sum_{j\in\mathcal{J}} y_j^t \leq z_0^t$ is equivalent by definition to $\sum_{j\in\mathcal{J}} \tilde{w}_j^t \leq z_0^t - \sum_{j\in\mathcal{J}} z_j^t$ where the right-hand side represents the number of units that are on-hand which are not already committed to a job after the demand in t is realized. Not requiring $\sum_{j\in\mathcal{J}} y_j^t$ to exactly equal z_0^t allows us to have backorders and unallocated on-hand inventory simultaneously, in light of z_0^t being defined as the net inventory and y_j^t being defined as the allocation level for j.

For example, if $J = 2$, it is possible to have z_0^t at -1, z_1^t at -5, z_2^t at -2 where we want to raise z_1^t to $y_1^t = 0$ and set $y_2^t = z_2^t = -2$. Here we start off with $z_0^t - (z_1^t + z_2^t) = -1 - (-5 - 2) = 6$ units of inventory after the arrival of the order placed a lead time ago and we have 5 units of backorder for jobs of type 1, and 2 units for jobs of type 2, after the demand for the current period is realized. We might allocate 5 units to jobs of type 1 and 0 to jobs of type 2 perhaps because it is desirable to have these backorders in the system for this time period.

Note that z^t evolves according to $z_j^{t+1} = y_j^t - \tilde{D}_t^j$ for $j = 1, ..., J$ and $z_0^{t+1} = z_0^t + q^{t+1} - \tilde{D}_0^t$; and Q^{t+1} is the vector $(q^{t+1}, ..., q^{t+L})$. We now apply the Clark-and-Scarf decomposition approach to solve Problem (2.21). In particular, Problem (2.21) is decomposed into a single-location inventory problem and J job-type problems. We start by defining these job-type problems and the single-location order problem. We will then relate them to Problem (2.21) above.

Define the terminal value function $v_{j,T'+1}$ to be 0 for $j = 1, ..., J$. The **job-type problems** are as follows:

$$v_{jt}(z) = \min_{z \le y \le 0} \{L_{jt}(y) + \mathbb{E} v_{j,t+1}(y - \tilde{D}_j^{t+1})\},\ z \le 0,\ j = 1, ..., J,\ 1 \le t \le T'. \quad (2.22)$$

Let $g_{jt}(y) = L_{jt}(y) + \mathbb{E} v_{j,t+1}(y - \tilde{D}_j^{t+1})$. Then

$$v_{jt}(z) = \min_{z \le y \le 0} g_{jt}(y). \quad (2.23)$$

Proposition 2.3. *Either $y = 0$ or $y = z$ is a minimizer for the minimization problem appearing in (2.22). As a result, $g_{jt}(y)$ and $v_{jt}(z)$ are affine in $y \in (-\infty, 0]$ and $z \in (-\infty, 0]$ respectively for all j and for all t.*

Proof: Assume as the induction hypothesis that $v_{j,t+1}$ is affine on $z \in (-\infty, 0]$. This is trivially true for $v_{t,T'+1}$. Because $v_{j,t+1}$ is affine on $(\infty, 0]$ by the induction hypothesis, $\mathbb{E} v_{j,t+1}(y - \tilde{D}_j^{t+1})$ is affine on $y \in (-\infty, 0]$. Since $L_{jt}(y) = \tilde{f}_j^t(y^-) - hy = (-\tilde{f}_j^t - h)y$ is linear in y on $(-\infty, 0]$, it is clear that $g_{jt}(y)$ is affine on $(-\infty, 0]$. Therefore, $\min_{z \le y \le 0} g_{jt}(y)$ has its minimum attained either at $y = 0$ or at $y = z$.

Furthermore, if $g_{jt}(z) > g_{jt}(0)$ for any value of $z < 0$, then $g(z) > g(0)$ for all values of $z < 0$. Consequently,

$$v_{jt}(z) = \begin{cases} g_{jt}(0), & \text{if } g_{jt}(\epsilon) > g_{jt}(0) \text{ for any } \epsilon < 0; \\ g_{jt}(z), & \text{otherwise,} \end{cases} \tag{2.24}$$

and v_{jt} is affine in z provided $z \leq 0$. This establishes the induction hypothesis and proves the proposition. $\square$

According to this proposition, we should either set the allocation level to zero for jobs of type j or we should allocate nothing so that the net inventory committed to jobs of type j remains at z_j^t. If the backorder cost per period is strictly positive, it is not optimal to keep backorders in the system when we have the option of satisfying them with on-hand inventory. But with the adjusted backorder cost $\tilde{f}_j^t$ in our subproblem being unrestricted in sign, it may be desirable to keep backorders in the system. This is when $y = z$ is the minimizer for the minimization problem appearing in (2.22). As discussed earlier, these could correspond to the reservation of inventory for a job type that has higher-priority demands anticipated to arrive in the near future, or the current unavailability of other service parts needed for job completion in the original multi-part-multi-job-type problem.

That the minimizer is either $y = 0$ or $y = z$ in the minimization problem appearing in (2.22) partitions the set $\mathcal{J}$ into two. We define $\mathcal{J}_t^*$ to be the subset of job types in $\mathcal{J}$ which are participating in the allocation problem. That is, $j \in \mathcal{J}_t^*$ if and only if the minimizer used to define v_{jt} is equal to 0. For $j \in \mathcal{J}_t^*$, it is desirable to satisfy all the backorders in period t after the demand is realized. We include in $\mathcal{J}_t^*$ also those job types whose objective functions appearing in (2.22) are constant over $z \leq y \leq 0$.

If $j \in \mathcal{J} \setminus \mathcal{J}_t^*$, it does not get allocated any inventory because it is desirable to have backorders of type j remain in the system in period t. Observe, in addition, the following corollary regarding the value functions for $j \in \mathcal{J} \setminus \mathcal{J}_t^*$ coming directly from the proof of Proposition 2.3:

Corollary 2.4. *For $j \in \mathcal{J} \setminus \mathcal{J}_t^*$, the value function $v_{jt}(z_j^t)$ decreases linearly to $-\infty$ as z_j^t decreases to $-\infty$.*

Consider the **single-part, multi-job-type allocation problem** represented by:

$$G_t(z^t) = \min_{\substack{z_j^t \le y_j \le 0,\ j \in \mathcal{J}, \\ \sum_{j \in \mathcal{J}} y_j \le z_0^t}} \left\{ \sum_{j \in \mathcal{J}} g_{jt}(y_j) \right\}. \tag{2.25}$$

We seek a lower bound for $G_t(z^t)$ that is separable by job type. In doing so, we make use of the following equivalent representation of $G_t(z^t)$:

Lemma 2.5. *Define $\mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)$ to be the set of y satisfying $z_j^t \le y_j \le 0,\ j \in \mathcal{J}_t^*$ and $\sum_{j \in \mathcal{J}_t^*} y_j \le z_0^t - \sum_{j \in \mathcal{J} \setminus \mathcal{J}_t^*} z_j^t$, we have*

$$G_t(z^t) = \sum_{j \in \mathcal{J}} v_{jt}(z_j^t) + \min_{y \in \mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)} \left\{ \sum_{j \in \mathcal{J}_t^*} [g_{jt}(y_j) - g_{jt}(0)] \right\}.$$

Proof: By Proposition 2.3, we know that

$$\sum_{j \in \mathcal{J}} v_{jt}(z_j^t) = \sum_{j \in \mathcal{J}_t^*} g_{jt}(0) + \sum_{j \in \mathcal{J} \setminus \mathcal{J}_t^*} g_{jt}(z_j^t). \tag{2.26}$$

Now consider the minimization problem which defines $G_t(z^t)$ in (2.25). With the constraints $z_j^t \le y_j \le 0,\ j \in \mathcal{J}$ and $\sum_{j \in \mathcal{J}} y_j \le z_0^t$, the optimal solution to (2.25) must have $y_j^* = z_j^t$ for $j \in \mathcal{J} \setminus \mathcal{J}_t^*$. Suppose this is not the case and $y_j^* = z_j^t + \epsilon \le 0$ for some $\epsilon > 0$ and some $j \in \mathcal{J} \setminus \mathcal{J}_t^*$. Since $j \in \mathcal{J} \setminus \mathcal{J}_t^*$, $v_{jt}(z)$ decreases linearly to $-\infty$ as $z \to -\infty$ by Corollary 2.4. Therefore, we can construct an alternate solution to the constraints of (2.25) where y_j is set to be equal to $z_j^t \leq z_j^t + \epsilon$. That we have another feasible solution which gives a strictly lower objective value contradicts the optimality of the given solution where $y_j^* = z_j^t + \epsilon \leq 0$ for some $\epsilon > 0$ and some $j \in \mathcal{J} \setminus \mathcal{J}_t^*$.

Therefore, we can set $y_j = z_j^t$ for $j \in \mathcal{J} \setminus \mathcal{J}_t^*$ in (2.25). Doing this, we get

$$\begin{aligned}
G_t(z^t) &= \min_{y \in \mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)} \left\{ \sum_{j \in \mathcal{J}_t^*} g_{jt}(y_j) \right\} + \sum_{j \in \mathcal{J} \setminus \mathcal{J}_t^*} g_{jt}(z_j^t) \\
&= \min_{y \in \mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)} \left\{ \sum_{j \in \mathcal{J}_t^*} [g_{jt}(y_j) - g_{jt}(0)] \right\} + \sum_{j \in \mathcal{J} \setminus \mathcal{J}_t^*} g_{jt}(z_j^t) + \sum_{j \in \mathcal{J}_t^*} g_{jt}(0) \\
&= \min_{y \in \mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)} \left\{ \sum_{j \in \mathcal{J}_t^*} [g_{jt}(y_j) - g_{jt}(0)] \right\} + \sum_{j \in \mathcal{J}} v_{jt}(z_j^t) \qquad (2.27)
\end{aligned}$$

where the second inequality followed by adding and subtracting $\sum_{j \in \mathcal{J}_t^*} g_{jt}(0)$ and the last inequality follows by (2.26). □

Note that the first term (2.27) is always positive because for $j \in \mathcal{J}_t^*$, $y_j = 0$ is the minimizer of $g_{jt}(y_j)$ over $y_j \leq 0$ and $y \in \mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)$ implies that $y_j \leq 0$ for $j \in \mathcal{J}_t^*$.

Consider the following penalty function

$$\Delta_{\mathcal{J}_t^*}(z_0^t) = \min_{\substack{y_j \leq 0, j \in \mathcal{J}_t^*, \\ \sum_{j \in \mathcal{J}_t^*} y_j \leq z_0^t}} \left\{ \sum_{j \in \mathcal{J}_t^*} [g_{jt}(y_j) - g_{jt}(0)] \right\}. \qquad (2.28)$$

In the special case that $\mathcal{J}_t^* = \mathcal{J}$ which is when all the job types participate in allocation, we have the result below which follows directly from Lemma 2.5 because $\{y_j \leq 0, j \in \mathcal{J}_t^*, \sum_{j \in \mathcal{J}_t^*} y_j \leq z_0^t\}$ is a relaxation of the feasible set $\mathcal{H}(y|z^t, \mathcal{J}, \mathcal{J}_t^*)$:

Corollary 2.6. *If $\mathcal{J}_t^* = \mathcal{J}$, we have the following lower bound for $G_t(z^t)$:*

$$G_t(z^t) \geq \Delta_{\mathcal{J}_t^*}(z_0^t) + \sum_{j \in \mathcal{J}} v_{jt}(z_t^j).$$

That is, $G_t(z^t)$ admits a separable lower bound consisting of a non-trivial penalty function whenever $\mathcal{J}_t^* = \mathcal{J}$. In general, we rely on the modified penalty function

$$\Delta_t(z_0^t) = \mathbb{1}_{\{\mathcal{J}_t^* = \mathcal{J}\}} \Delta_{\mathcal{J}}(z_0^t). \tag{2.29}$$

Corollary 2.7. *We have the following lower bound for $G_t(z^t)$:*

$$G_t(z^t) \geq \Delta_t(z_0^t) + \sum_{j \in \mathcal{J}} v_{jt}(z_t^j).$$

With $\Delta_t(z_0^t) = 0$ whenever $\mathcal{J}_t^* \neq \mathcal{J}$, this is a much looser lower bound than the special case where $\mathcal{J}_t^* = \mathcal{J}$, and so its value in practice will depend on how frequently this special situation arises.

Using the Clark-and-Scarf decomposition approach, one would solve the job-type problems in (2.22) to determine the allocate-up-to backorder level for each of the job types. In deciding how much inventory to purchase, one considers a single-location inventory problem where the one-period cost function involves L_{0t} as defined in (2.20). In addition, one incorporates into the one-period cost function the penalty function defined in (2.29) for cases when the net-inventory is not enough to cover the desired allocate-up-to backorder levels of all the job types.

In the original single-part-multi-job-type problem, procurement decisions are made up to period T and allocation decisions are made up to T'. The one-period cost function appearing in the single-location inventory problem corresponds to the expected cost incurred at the end of the period a lead time later.

If we let $v_{0,T+1}$ be the terminal function for the single-location inventory problem, it needs to count the cost incurred between $T+1+L$ and T' for an exact comparison with the original single-part-multi-job-type problem. Therefore, assuming $\tilde{z}$ to be the inventory position at the beginning of $T+1$, the terminal value function is

$$v_{0,T+1}(\tilde{z}) = \sum_{s=T+1+L}^{T'} \mathbb{E}L_{0s}\left(\tilde{z}\right) + \sum_{s=T+1+L}^{T'} \mathbb{E}\Delta_s(\tilde{z}). \tag{2.30}$$

With this boundary condition defined, we formulate the following **single-location inventory problem** which can be used to compute the order quantity for the service part under consideration:

$$\begin{aligned} v_{0t}(\tilde{z}) &= \min_{y \geq \tilde{z}} c(y-\tilde{z}) + \mathbb{E}L_{0,t+L}\left(y - \sum_{u=t+1}^{t+L} \tilde{D}_0^u\right) \\ &\quad + \mathbb{E}\Delta_{t+L}(y - \sum_{s=t+1}^{t+L} \tilde{D}_0^s) + \mathbb{E}v_{0,t+1}(y - \tilde{D}_0^{t+1}),\ 1 \leq t \leq T. \end{aligned} \tag{2.31}$$

As in the terminal function (2.30), the argument $\tilde{z}$ should be interpreted as an inventory position which corresponds to the net-inventory plus the on-order inventory. The decision variable y appearing in the minimization problem should be interpreted as an order-up-to inventory position level. If we order up to y at time t after the realization of $\tilde{D}_0^t$, the net-inventory at the beginning of period $t+L$ after the realization of $\tilde{D}_0^{t+L}$ is equal to $y - \sum_{u=t+1}^{t+L} \tilde{D}_0^u$. Formulation (2.31) can be used to obtain the desired order quantity for the service part under consideration.

Before we show how the single-location inventory problem and the job-type problems relate to the value function of the modified single-part-multi-job-type problem, we define the function $F_{0t}(z_0^t, Q^t)$ to keep track of the unavoidable cost

L_{0t} incurred between t and $t+L-1$ given state variables z_0^t and Q^t:

$$F_{0t}(z_0^t, Q^t) = \sum_{s=t}^{t+L-1} \mathbb{E}L_{0s}\left(z_0^t + \sum_{u=t+1}^{s} q^u - \sum_{u=t+1}^{s} \tilde{D}_0^u\right). \tag{2.32}$$

This cost is unavoidable because the random net-inventory $z_0^t + \sum_{u=t+1}^{s} q^u - \sum_{u=t+1}^{s} \tilde{D}_0^u$ in period s, where $t \leq s \leq t+L-1$, cannot be altered by the decision made at time t. Together with another term that keeps track of the unavoidable penalty incurred between t and $t+L-1$, the Clark and Scarf decomposition for $1 \leq t \leq T$ is given below:

$$\begin{aligned}\tilde{V}_t(z^t, Q^t) = F_{0t}(z_0^t, Q^t) &+ \sum_{s=t}^{t+L-1} \mathbb{E}\Delta_s(z_0^t + \sum_{u=t+1}^{s} q^u - \sum_{u=t+1}^{s} \tilde{D}_0^u) \\ &+ v_{0t}(z_0^t + \sum_{s=t+1}^{t+L-1} q^s) + \sum_{j \in \mathcal{J}} v_{jt}(z_j^t) \qquad (2.33) \\ &\stackrel{\text{def}}{=} \tilde{v}_{0t}(z_0^t, Q^t) + \sum_{j=1}^{J} v_{jt}(z_j^t). \qquad (2.34)\end{aligned}$$

In general, the decomposition provides a lower bound for the value function (2.21) of the modified single-part-multi-job-type problem:

Proposition 2.8. $\tilde{V}_t(z^t, Q^t) \leq V_t(z^t, Q^t)$.

Proof: This can be shown using induction. Because the notation becomes very cumbersome with a general lead time and no insights are gained by considering the general case, we fix L to be equal to 1 here. With $L = 1$, Q^t is an empty set and we drop its appearance wherever applicable. Also note that when $L = 1$, the net inventory z_0^t is also the inventory position before the order q^{t+1} is placed at time t. The result of the proposition can be easily shown for $t = T$, and assuming that we have the result for $t+1$, we immediately get that

$$\begin{aligned}
V_t(z_t) &= \min_{\substack{z_j^t \le y_j^t \le 0,\ j\in\mathcal{J},\\ q^{t+L}\ge 0,\\ \sum_{j\in\mathcal{J}} y_j^t \le z_0^t}} \left\{ cq^{t+1} + L_{0t}(z_0^t) + \sum_{j=1}^{J} L_{jt}(y_j^t) + \mathbb{E}V_{t+1}\left(z^{t+1}\right) \right\} \\
&\ge \min_{\substack{z_j^t \le y_j^t \le 0,\ j\in\mathcal{J},\\ y\ge z_0^t,\\ \sum_{j\in\mathcal{J}} y_j^t \le z_0^t}} \left\{ c(y - z_0^t) + L_{0t}(z_0^t) + \sum_{j=1}^{J} L_{jt}(y_j^t) \right. \\
&\qquad \left. + \mathbb{E}\tilde{v}_{0,t+1}(y - \tilde{D}_0^{t+1}) + \sum_{j\in\mathcal{J}} \mathbb{E}v_{j,t+1}(y_j^t - \tilde{D}_j^t) \right\}.
\end{aligned}$$

Based on the definition of $\tilde{v}_{0t}$ given in (2.34), $\mathbb{E}\tilde{v}_{0,t+1}(z_0^{t+1}) = \mathbb{E}L_{0,t+1}(z_0^{t+1}) + \mathbb{E}\Delta(z_0^{t+1}) + \mathbb{E}v_{0,t+1}(z_0^{t+1})$. Substituting this in and splitting the optimization problem above where there are no interactions among the decision variables, we get that:

$$\begin{aligned}
V_t(z^t) &\ge L_0^t(z_0^t) + \min_{y\ge z_0^t} \left\{ c(y - z_0^{t+1}) + \mathbb{E}L_{0,t+1}(y - \tilde{D}_0^{t+1}) \right. \\
&\qquad \left. + E\Delta_{t+1}(y - \tilde{D}_0^{t+1}) + \mathbb{E}v_{0,t+1}(y - \tilde{D}_0^{t+1}) \right\} \\
&\qquad + \min_{\substack{z_j^t \le y_j^t \le 0,\ j\in\mathcal{J}\\ \sum_{j\in\mathcal{J}} y_j^t \le z_0^t}} \left\{ \sum_{j\in\mathcal{J}} \left(L_{jt}(y_j^t) + \mathbb{E}v_{j,t+1}(y_j^t - \tilde{D}_t^j) \right) \right\} \\
&= L_0^t(z_0^t) + v_{0t}(z_0^t) + \min_{\substack{z_j^t \le y_j^t \le 0,\ j\in\mathcal{J}\\ \sum_{j\in\mathcal{J}} y_j^t \le z_0^t}} \left\{ \sum_{j\in\mathcal{J}} \left(L_{jt}(y_j^t) + \mathbb{E}v_{j,t+1}(y_j^t - \tilde{D}_t^j) \right) \right\} \\
&= L_0^t(z_0^t) + v_{0t}(z_0^t) + G_t(z^t)
\end{aligned}$$

So we have recovered $F_{0t}(z_0^t) = L_0^t(z_0^t)$ and $v_{0t}(z_0^t)$. Using Corollary 2.7, we may now conclude that

$$V_t(z^t) \ge F_{0t}(z_0^t) + v_{0t}(z_0^t) + \Delta_t(z_0^t) + \sum_{1\le j\le J} v_{jt}(z_j^t) = \tilde{V}_t(z^t).$$

□

This result allows us to solve a lower-bound approximation to the modified single-part-multi-job-type problem in a computationally tractable manner. Instead of dealing with the multi-dimensional state space, we may now decompose the single-part-multi-job-type problem further into a single-location inventory problem (2.31) and J job-type problems (2.22). Each of these problems employs a one-dimensional state space.

Finally, as discussed earlier, our definition of $\Delta_t(\cdot)$ makes it non-zero only if $\mathcal{J}_t^* = \mathcal{J}$. If any job type exhibits a negative backordering cost after the dual variables from the linear program solution have been incorporated, the penalty term will be equal to zero. A negative backordering cost for a particular job type economically indicates the temporary infeasibility of allocating to this job type any on-hand inventory in view of what is currently on-hand and on-order as well as in anticipation of what may be arriving in the future. It is not hard to imagine cases in which this bound becomes too loose and leads to a procurement quantity which is too low despite perhaps not even having enough inventory in the system (on-hand and on-order) to satisfy all outstanding orders. In our numerical experiments, we also consider obtaining order quantities via the following dynamic program:

$$\begin{aligned} v_{0t}(\tilde{z}) &= \min_{y \geq \tilde{z}} c(y - \tilde{z}) + \mathbb{E}L_{0,t+L}\left(y - \sum_{u=t+1}^{t+L} \tilde{D}_0^u\right) \\ &\quad + \mathbb{E}\Delta_{\mathcal{J}_{t+L}^*}(y - \sum_{s=t+1}^{t+L} \tilde{D}_0^s) + \mathbb{E}v_{0,t+1}(y - \tilde{D}_0^{t+1}),\ 1 \leq t \leq T. \end{aligned} \tag{2.35}$$

The function $\Delta_{\mathcal{J}_{t+L}^*}$ was defined in (2.28) and it takes into account only those job types that participate in allocation. For solutions derived in this manner, however, we cannot claim that they correspond to a lower bound on the value functions of the modified single-part-multi-job-type problem.

2.8 Numerical Experiments

In this section, we report numerical results that illustrate the performance of our policy. For comparison purposes, we define a benchmark policy in terms of a benchmark procurement policy and a benchmark allocation policy. The benchmark procurement policy optimizes the order-up-to level for each service part under the assumption that the demand streams for the different service parts are independent. The benchmark allocation rule is a myopic allocation rule in which Equation (2.3) is solved under the assumption that the future value function is zero, $V_{t+1} = 0$. We describe the benchmark procurement rule in detail in the following subsection.

2.8.1 The Benchmark Procurement Rule

We now describe the benchmark independent order-up-to procurement rule. By independent, we mean that the policy will assume the arrival processes of the different job types to be independent. Imputed part-based backordering costs will be calculated. The service parts will be managed independently of each other and the order-up-to level for a service part will depend solely on its imputed inventory positions. Consider part $i \in \mathcal{I}$, we define its imputed backordering cost per period as:

$$\pi_i = \frac{1}{T} \sum_{t=1}^{T} \left(\sum_{j \in \mathcal{J} : r_{ij} > 0} \left[\frac{\mathbb{E}[D_j^t] r_{ij}}{\sum_{j \in \mathcal{J}} \mathbb{E}[D_j^t] r_{ij}} \right] \left(\frac{\tilde{f}_j^t}{r_{ij}} \right) \right) \tag{2.36}$$

This is similar to that defined in Lu and Song (2005) when they considered the item-based optimization problem as an approximation. The imputed demand for part i in period t is defined as

$$\tilde{D}_i^t = \sum_{j \in \mathcal{J}} r_{ij} D_j^t. \tag{2.37}$$

We consider two types of demand distributions in this chapter: Poisson and negative binomial. In the case that D_j^t has a Poisson distribution with mean ν_j^t, we approximate $\tilde{D}_i^t$ using a Poisson distribution with mean $\sum_{j \in \mathcal{J}} r_{ij} \nu_j^t$. In the case that D_j^t has a negative binomial distribution with mean ν_j^t, we assume that they all have the same underlying variance to mean ratios and we approximate $\tilde{D}_i^t$ using a negative binomial distribution with mean $\sum_{j \in \mathcal{J}} r_{ij} \nu_j^t$ and the same variance to mean ratio as each of the D_j^t. The Poisson and negative binomial assumptions allow us to easily estimate the lead time demand for each of the service parts because the sum of independent Poisson variables still has a Poisson distribution and the sum of independent negative binomial variables still has a negative binomial distribution provided they share the same variance to mean ratio.

Consider the standard single-part-single-demand-stream subproblem for service part $i \in \mathcal{I}$. Given state vector (x^t, D^t, B^t, Q^t) following the arrival of jobs in time period t, the imputed inventory position for part i is equal to

$$\tilde{z}_i^t = x_i^t + \sum_{s=1}^{L_i - 1} q_i^{t+s} - \sum_{j \in \mathcal{J}} r_{ij} D_j^t - \sum_{j \in \mathcal{J}} r_{ij} b_j^t. \tag{2.38}$$

The following dynamic program for $1 \leq t \leq T$ with terminal condition $V_{T+1}^i(\cdot) = 0$ is solved to obtain the independent order-up-to level for service part $i \in \mathcal{I}$:

$$\begin{aligned} V_t^i(\tilde{z}_i^t) = \min_{y_i^t \geq \tilde{z}_i^t} \Bigg\{ & c(y_i^t - \tilde{z}_i^t) + h_i \mathbb{E} \left(y_i^t - \sum_{s=t+1}^{t+L_i} \tilde{D}_i^t \right)^+ \\ & + \pi_i \mathbb{E} \left(\sum_{s=t+1}^{t+L_i} \tilde{D}_i^t - y_i^t \right)^+ + \mathbb{E} V_{t+1}^i (y_i^t - \tilde{D}_i^{t+1}) \Bigg\}. \end{aligned} \tag{2.39}$$

The order-up-to levels obtained using this formulation are the benchmark procurement quantities for part $i \in \mathcal{I}$ in each period. This is the benchmark procurement policy.

2.8.2 A Canonical Example: Two Service Parts and Two Job Types

To gain insights into some algorithms, it is typical to begin analysis with canonical systems. See, for example, the "W" system involving three service parts and two job types in the work of Dogru et al. (2010). We consider in this section the "N" system that consists of two job types and two service parts. A service part of type 1 is required by both job types (1 and 2), and a service part of type 2 is required only by a job of type 2. The figure below illustrates this system.

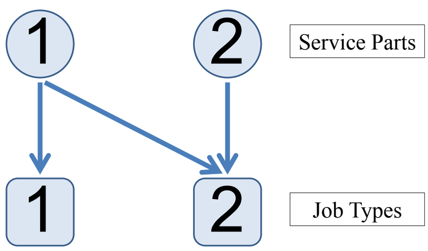

Figure 2.1: This figure illustrates our canonical example: an "N" system with two service parts and two job types.

In terms of allocation, we have in section 2.5 and earlier in this section described the price-driven and myopic approaches. In terms of procurement, we described the independent order-up-to approach in the previous subsection and we described in sections 2.6 and 2.7 the Clark and Scarf (C&S) decomposition approach. At the end of section 2.7, we noted a version of Clark and Scarf decomposition where job types with negative backordering costs are removed from consideration in determining the procurement quantities. The policies that we compare have their characteristics summarized in Table (2.1). We will refer to these policies by their acronyms.

Policy	Allocation	Procurement
MYBS	myopic	independent order-up-to
PDBS	price-driven	independent order-up-to
PDCS	price-driven	C&S decomposition
PDCS(*)	price-driven	C&S decomposition excluding negative-cost job types

Table 2.1: Description of the tested policies

Where dual variables were employed to drive allocation and procurement decisions, we sampled future demand trajectories, solved multiple linear programs and took the average of the dual values obtained, as discussed in section 2.5. In all cases, each set of linear programs consisted of 50 randomly generated samples. It is a postulate that better results may be obtained using multiple LPs due to Proposition 2.1.

The performance of our policy was tested extensively over the parameter space. We group the simulations into three general sets. In the first set, the per-period demand for each of the job types follows a Poisson distribution and is stationary over time. In the second set, the per-period demand still follows a Poisson distribution but it varies over time in a periodic manner. In the last set, it follows a negative binomial distribution. A base case was selected in each of the three sets.

Table 2.2 summarizes the parameters used for the three base cases. The time horizon has 32 periods for all of the simulations. Holding costs, h_1 and h_2, for the two service parts are both set equal to five. The lead time, L_1, of the common component is set to six, which is twice the lead time, L_2, of the unique component. Job type 2 is the more complex assembly as it requires both service parts. The backorder cost, f_2, of the complex assembly is 20, which is twice the backorder cost, f_1, of the simple assembly. These numbers are chosen to create a situation in which a coordinated policy is likely to outperform the benchmark policy.

In the stationary Poisson demand set, the mean of the one-period demand is 4 for the more complex job type and 2 for the other. The base case in the periodic Poisson demand set has the same parameters except every four periods, the mean of the per-period demand doubles for both job types. As for the base case in the stationary negative binomial demand set, the values of the parameters are also chosen to be the same as for the Poisson case, with the additional parameter, the coefficient of variation, set at 1 for the more complex job type. Recall that to ensure that demands over lead times for the parts convolute easily, all the negative binomial demand distributions must have the same variance to mean ratio. Hence, once the coefficient of variation parameter is specified for the more complex job type, it is also completely determined for the other job type (1.4, in this case).

From each of the base cases, we generate other parameter sets as follows. Along any one of the dimensions of the parameter space, we put the base case in the "center" and go in both directions changing the value of one specific parameter. For example, in testing the effect of the procurement lead time of the

Demands	Stationary Poisson	Periodic Poisson	Stationary Neg. Bin.
(h_1, h_2)	(5,5)	(5,5)	(5,5)
(L_1, L_2)	(6,3)	(6,3)	(6,3)
(f_1, f_2)	(10,20)	(10,20)	(10,20)
$\mathbb{E}[\vec{D}_t]$	(2,4)	(2,4) / (4,8) *	(2,4)
Coeff. of Var. of $\vec{D}_t$	N/A	N/A	(1.4,1)

Table 2.2: Parameters used for the base-case experiments. *Note: For the periodic case, the demands change every four periods and remain constant for four periods.

common part, we look at the range of values between 3 and 10. The base cases have this parameter set at 6. The three adjacent tables for any single dimension of the parameter space correspond to the three different types of general demand characteristics mentioned earlier: stationary Poisson, periodic Poisson and stationary negative binomial. The dimensions we considered include the holding cost of the parts, the procurement lead time of the common part, the mean of the demand of the more complex job type, as well as the backorder cost of the more complex job type. Eight distinct values are considered for each of these dimensions. With 3 general demand types, 4 dimensions of the parameter space, 8 distinct values for each of the dimensions, there are a total of 96 test cases. For the general demand type using negative binomial distributions, we also varied the coefficient of variation of the more complex job. There are 8 additional test cases considered along this dimension.

We determined that the price-driven Clark-and-Scarf policy outperformed the benchmark policy in most cases. These details are summarized in Table (2.3) to Table (2.7) at the end of this chapter. Where the holding cost was varied, it was changed for both the unique and the common service part. All other parameters took their base-case values. It was found along the holding cost dimension of the parameter space that the PDCS (prive-driven allocation, Clark-and-Scarf procurement) policy outperformed the benchmark MYBS (myopic allocation, independent procurement) policy by between 2.79% and 7.79% where a statistically significant difference was observed for the case with stationary Poisson demands. These performance gaps were between 3.18% and 8.47% for the periodic Poisson case and between 3.12% and 7.27% for the negative binomial case. Among the 24 test cases along the holding cost dimension, MYBS and PDCS had no statistically significant performance gaps in three different instances. There is no apparent pattern that would explain when the gap will be significant and when it will not be.

Where procurement lead times were varied, only the lead time of the common part was changed. Two out of 24 instances in this dimension displayed no statistically significant difference between PDCS and MYBS. Both of these instances had negative binomial demand distributions but the procurement lead times used in these instances are not at the end points of the range of values considered. Among the rest of the test cases, PDCS outperformed MYBS by between 2.18% and 7.47% for the stationary Poisson case, by between 3.79% and 8.88% for the periodic Poisson case, and by between 4.10% and 5.26% for the negative binomial case.

In changing the mean of the job type demand, we kept the mean of the simpler job fixed while varying that of the more complex job type. The mean of the simpler job was set at 2 while the mean of the more complex job varied between 1 and 8. For the case with stationary Poisson demands, PDCS outperformed MYBS by between 3.73% and 6.28%. This range is between 3.15% and 7.27% for the periodic Poisson case and between 2.04% and 6.73% for the negative binomial case. Statistically insignificant differences were observed in two instances, one instance has its stationary Poisson mean set at 1 and the other instance has its stationary negative binomial mean set at 3.

When we tested the impact of backorder costs, we kept the backorder cost of the simpler job fixed while varying that of the more complex job. We kept the backorder cost of the simpler job at 10 per backorder per period while we varied the backorder cost of the more complex job between 10 and 27.5 per backorder per period. In the stationary Poisson case where the backorder cost of the more complex job was set equal to that of the simpler job at 10 and the negative binomial cases where the backorder costs of the more complex job were set to 10 and 12.5, PDCS had no statistically significant difference from MYBS. Among the other test cases, PDCS outperformed MYBS by between 4.54% and 6.15% under stationary Poisson demands, by between 2.26% and 8.54% under periodic Poisson demands and by between 2.88% and 5.26% under negative binomial demands. Finally, when we varied the coefficient of variation between 0.6 and 1.2 under negative binomial demands, PDCS outperformed MYBS in every test case and the performance gap ranges between 3.96% and 5.82%.

While it cannot be said in complete generality in what parameter scenarios the PDCS policy outperformed the MYBS policy, the values of the various parameters which amplified the cost benefit of our policy agreed with our initial intuition to some extent. There was some tendency for the gap between the total costs incurred under the PDCS policy and the benchmark MYBS policy to widen as the value of the parameter considered along a given direction increased. We exemplify this using the plot in Figure (2.2) which is based on the MYBS and PDCS data obtained under stationary Poisson demands where the procurement lead time of the common part was varied.

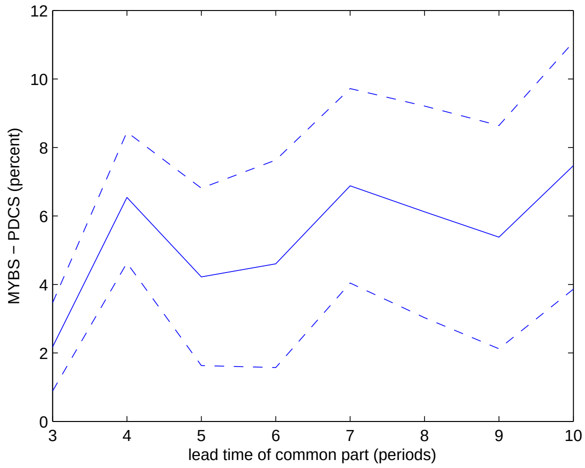

Figure 2.2: Plot of the percentage difference between MYBS and PDCS under stationary Poisson demands where the lead time of the common part is varied. The dotted lines indicate where the 95% confidence interval is.

In addition to PDCS, we also tested the algorithm PDCS(*) on all cases derived from the "N" system. We found that there is essentially no statistical difference between the adjusted algorithm PDCS(*) and the PDCS algorithm for the canonical "N" system.

2.8.3 Larger Problems

The PDCS (price-driven allocation, Clark-and-Scarf procurement) algorithm scales well in solving larger problems. Given a problem with T periods where the lengthiest lead time is $\bar{L}$, m service parts and n job types, each deterministic linear program that we have to solve in period 1 consists of $(m+n)(T+\bar{L})$ decision variables and $(m+n)(T+\bar{L})$ inventory and outstanding orders balance constraints. In doing the nonlinear approximation, we solve m dynamic programs (each with a single-dimension state space) in order to obtain the procurement quantities. Hence, this algorithm is not affected by the curse of dimensionality that arises in traditional dynamic programming.

In testing our algorithm on larger problems, we varied the number of service parts and the number of job types involved. We varied the number of service parts m between 3 and 6 and fixed the corresponding number of job types at $m - 1$. The bill-of-materials matrices have their two diagonals (since these matrices are m by $m - 1$) set as 1 and everything else is 0. In other words, any two consecutive job types (in terms of job type numbers) have one overlapping service part. Finally, we tested a few cases in which there are 6 service parts and 15 job types involved, matching approximately the number of service parts for which inventory needs to be managed carefully in realistic equipment overhaul situations. Of these 6 service parts, two are "either-or" parts meaning all job types require one of them. The rest of the bill-of-materials matrix is filled in such a way that the 15 columns (representing the 15 job types) consist of all possible combinations of zeros and ones except for a column with all zeros. The number of periods was set at 32 for all cases just like before.

In these larger problems, allocation becomes more difficult to optimize due to the complex dynamics and correlation structure involved. We found that negative backorder costs frequently arise in these problems. It turns out that it made a substantial difference whether we used PDCS or PDCS(*) (the version of PDCS where all the job types with negative backorder costs are removed in determining the order quantities). Where PDCS gave rise to results which were either no better than or worse than those of MYBS, PDCS(*) yielded noticeable cost advantages. Where statistically significant, the cost advantage lies some-where between 1 and 4%. The detailed results, together with the values of the parameters, are included at the end of this chapter in Table (2.8) and Table (2.9). There was one problem instance in which the MYBS policy outperformed both PDCS and PDCS(*). Doubling the number of linear programs solved in each instant decreased the gap between the results, but the MYBS policy was still better in this example. Further investigation revealed that the PDCS(*) policy still opted for low procurement quantities when there was not enough inventory in the system to satisfy all outstanding orders. The objective function minimized in determining the procurement quantity was particularly flat in these instances between negative infinity and some small positive quantity. By treating numerical differences up to the order of 10^{-3} as negligible, we managed to modify PDCS(*) in a way that closed the gap completely.

Looking at the results detailed in the last section of this chapter, one may be inclined to draw the conclusion that a price-driven allocation method has no advantage over a myopic allocation method. However, the statistical differences observed between MYBS and PDCS/PDCS(*) need to be understood in the context that our procurement policy takes into account the correlated demands among the different service parts and, in general, orders smaller quantities than the simple independent order-up-to policy. This means that the benchmark procurement policy typically has higher inventory levels. Where inventory is plentiful, it is less necessary to allocate on-hand inventory carefully.

2.9 Concluding Remarks

In this chapter, we proposed the use of a novel algorithm in equipment overhaul problems where the demands for service parts are coupled. This finite-horizon problem has the same structure as that of assemble-to-order (ATO) systems. Without assuming the form of the allocation and procurement policies, we derived linear and non-linear approximations of the value function appearing in the dynamic program formulation. The resulting algorithm provides an alternate way of tackling these problems with arbitrary sizes if one does not want to settle with simple procurement strategies that go along with a pre-determined allocation method such as first-come-first-served. To our knowledge, this is the first time an algorithm with a non-standard allocation rule and a correlation-sensitive procurement strategy is put forth for a general ATO problem. The numerical experiments we ran indicated that this algorithm is a viable alternative to consider in managing inventory in an equipment overhaul setting. Even a 1% cost advantage translates to a large dollar amount in settings where the equipment parts are very costly. The idea of our proposed algorithm is simple and it also comes with nice economical interpretations.

From a computational point of view, it is worth noting that the way our algorithm works allows the employment of parallel processing to reduce computational time. The many generated linear programs can be solved simultaneously, provided that the master-client communication does not become detrimental. When many service parts are involved, it is also possible to solve the multiple part-type procurement problems simultaneously. From a testing standpoint, of course it is also possible to run the iterations on different processors. Further experimentation may be carried out to understand how our algorithm performs in other areas of the parameter space. While the bill-of-materials tested in this chapter were all assumed to be binary, our algorithm is not restricted at all by this assumption. It is also an easy extension to use our algorithm in cases where capacity constraints may be present.

From a performance standpoint, algorithms that are derived from lower-bound approximations generate lower bounds which we may use for comparison with the average cost incurred under our policy. Nonetheless, the lower bounds obtained in this chapter are not useful because the linear approximation of the value functions are derived using deterministic linear programs in which procurement and allocation decision variables are set in a manner that incurs no cost beyond the largest lead time (aside from the cost of purchasing service parts). The lower bounds we generate are, therefore, not helpful in determining how well our algorithm does as compared with the unknown optimal strategy.

It is worth pointing out that allocation decisions generally have short-term impact while procurement decisions have long-term impact. This is because an allocation decision is primarily concerned with how on-hand and on-order inventory should be distributed among outstanding jobs, while a procurement decision does not affect our system until a lead time later. That the deterministic linear programs set decision variables in a manner that incurs no holding and backorder cost beyond the maximum lead time also sheds light on why an approximation driven solely by linear programs gives no guidance on how much to order since such decisions only impact the system beyond the maximum procurement lead time.

In the work of Dogru et al. (2010) identical lead times are assumed in a simple “W” system consisting of three service parts and two job types. Solving this as an infinite-horizon average cost minimization problem, the value function is bounded below by a two-stage stochastic program with recourse where the two stages are separated by a lead time. As mentioned in the literature review section, this lower bound is attained by a policy inspired by the two-stage stochastic program. In light of this, it may be worthwhile to compare the lower bounds we obtained with the average cost incurred over the maximum lead time if we treat this as an infinite-horizon problem.

In real-life situations, backorder costs may not start accumulating when a piece of equipment arrives at a job shop for repair. A time window is usually present whose duration corresponds to the amount of the time a customer is willing to wait. From a modeling perspective, this may be captured using a non-linear backordering cost function for each job type (as a function of the amount of time spent by a job in the job shop). This assumption adds many more dimensions to the state space we currently have. Without assuming the form of the procurement policy, we end up, at the non-linear approximation level, with non-standard single-part-multi-job-type problems having non-linear backorder costs. The next two chapters will deal with inventory problems where the per-period cost of a backorder increases with age.

2.10 Appendix for Chapter 2 with Tables of Numerical Results

	Set 1		Set 2		Set 3	
H	MYBS - PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS
1	$0.22\% \odot$	$2.98\% \checkmark$	$0.75\% \checkmark$	$3.18\% \checkmark$	$0.30\% \checkmark$	$3.12\% \checkmark$
2	$-0.08\% \odot$	$2.79\% \checkmark$	$0.00\% \odot$	$3.87\% \checkmark$	$-0.10\% \odot$	$1.96\% \odot$
3	$-0.21\% \odot$	$2.36\% \odot$	$-0.47\% \odot$	$3.94\% \checkmark$	$0.18\% \odot$	$4.54\% \checkmark$
4	$0.09\% \odot$	$5.02\% \checkmark$	$-0.12\% \odot$	$6.51\% \checkmark$	$0.49\% \odot$	$3.72\% \checkmark$
5	$0.28\% \odot$	$4.60\% \checkmark$	$-0.46\% \odot$	$4.71\% \checkmark$	$0.50\% \checkmark$	$5.26\% \checkmark$
6	$-0.20\% \odot$	$5.87\% \checkmark$	$0.00\% \odot$	$8.47\% \checkmark$	$0.27\% \odot$	$7.27\% \checkmark$
7	$-0.26\% \odot$	$4.96\% \checkmark$	$0.00\% \odot$	$6.74\% \checkmark$	$0.70\% \checkmark$	$6.77\% \checkmark$
8	$-0.14\% \odot$	$7.79\% \checkmark$	$-0.18\% \odot$	$5.96\% \checkmark$	$0.19\% \odot$	$2.93\% \odot$

Table 2.3: Impact of holding costs on the performance of the PDCS algorithm. Listed here are the averages of our simulation results. The holding costs of both service parts were changed together. All other parameters took their base-case values. A check-mark indicates a statistically significant superiority at the 95% level and a cross indicates the opposite. A dot indicates that the difference is not statistically significant.

	Set 1		Set 2		Set 3	
L	MYBS - PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS
(3,3)	-0.39% $\odot$	2.18% ✓	-0.43% $\odot$	3.79% ✓	-0.26% $\otimes$	4.10% ✓
(4,3)	-0.46% $\otimes$	6.54% ✓	-0.20% $\odot$	5.28% ✓	-0.13% $\odot$	5.21% ✓
(5,3)	0.03% $\odot$	4.22% ✓	-0.45% $\odot$	4.23% ✓	0.01% $\odot$	4.21% ✓
(6,3)	0.28% $\odot$	4.60% ✓	-0.46% $\odot$	4.71% ✓	0.50% ✓	5.26% ✓
(7,3)	-0.05% $\odot$	6.88% ✓	0.41% $\odot$	8.88% ✓	0.58% ✓	3.47% $\odot$
(8,3)	-0.03% $\odot$	6.12% ✓	0.63% ✓	6.07% ✓	0.84% ✓	3.73% $\odot$
(9,3)	0.08% $\odot$	5.38% ✓	1.32% ✓	7.86% ✓	1.21% ✓	4.76% ✓
(10,3)	-0.19% $\odot$	7.47% ✓	1.45% ✓	4.39% ✓	0.73% $\odot$	4.88% ✓

Table 2.4: Impact of procurement lead times on the performance of the PDCS algorithm. Listed here are the averages of our simulation results. Only the lead time of the common part was varied. All other parameters took their base-case values. A check-mark indicates a statistically significant superiority at the 95% level and a cross indicates the opposite. A dot indicates that the difference is not statistically significant.

	Set 1		Set 2		Set 3	
$\mathbb{E}[D_t]$	MYBS - PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS
(2,1)	-0.37% $\odot$	0.30% $\odot$	-0.50% $\otimes$	3.15% ✓	-0.13% $\odot$	2.04% ✓
(2,2)	-0.19% $\odot$	3.89% ✓	-0.46% $\otimes$	5.31% ✓	-0.03% $\odot$	3.25% ✓
(2,3)	-0.10% $\odot$	5.05% ✓	-0.36% $\odot$	5.36% ✓	0.38% ✓	2.23% $\odot$
(2,4)	0.28% $\odot$	4.60% ✓	-0.46% $\odot$	4.71% ✓	0.50% ✓	5.26% ✓
(2,5)	-0.30% $\odot$	6.28% ✓	0.28% $\odot$	7.27% ✓	0.19% $\odot$	6.24% ✓
(2,6)	0.21% $\odot$	4.25% ✓	0.57% ✓	6.99% ✓	0.48% ✓	6.73% ✓
(2,7)	0.45% ✓	3.90% ✓	0.42% ✓	7.17% ✓	0.40% $\odot$	6.12% ✓
(2,8)	-0.07% $\odot$	3.73% ✓	0.32% $\odot$	6.26% ✓	0.54% $\odot$	5.13% ✓

Table 2.5: Impact of demand means on the performance of the PDCS algorithm. Listed here are the averages of our simulation results. Only the means of the more complex job type (the one requiring both service parts) were varied. All other parameters took their base-case values. A check-mark indicates a statistically significant superiority at the 95% level and a cross indicates the opposite. A dot indicates that the difference is not statistically significant.

	Set 1		Set 2		Set 3	
F	MYBS - PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS	MYBS -PDBS	MYBS -PDCS
(10,10)	-0.08% $\odot$	1.41% $\odot$	-0.43% $\odot$	2.26% ✓	-0.37% $\otimes$	1.39% $\odot$
(10,12.5)	-0.18% $\odot$	5.07% ✓	-0.16% $\odot$	5.85% ✓	0.05% $\odot$	1.05% $\odot$
(10,15)	-0.22% $\odot$	5.03% ✓	-0.38% $\odot$	3.27% ✓	-0.23% $\odot$	2.88% ✓
(10,17.5)	-0.22% $\odot$	5.45% ✓	-0.06% $\odot$	4.22% ✓	0.25% ✓	3.90% ✓
(10,20)	0.28% $\odot$	4.60% ✓	-0.46% $\odot$	4.71% ✓	0.50% ✓	5.26% ✓
(10,22.5)	0.16% $\odot$	6.15% ✓	0.27% $\odot$	4.55% ✓	0.31% $\odot$	4.83% ✓
(10,25)	0.04% $\odot$	4.54% ✓	0.43% ✓	8.54% ✓	1.13% ✓	3.72% ✓
(10,27.5)	0.53% ✓	5.34% ✓	0.44% $\odot$	7.46% ✓	0.78% $\odot$	4.86% ✓

Table 2.6: Impact of backordering costs on the performance of the PDCS algorithm. Listed here are the averages of our simulation results. Only the backordering cost of the more complex job type (the one requiring both service parts) was varied. All other parameters took their base-case values. A check-mark indicates a statistically significant superiority at the 95% level and a cross indicates the opposite. A dot indicates that the difference is not statistically significant.

	Set 3	
Coeff. of var.	MYBS - PDBS	MYBS - PDCS
0.6	0.16% $\odot$	4.25% ✓
0.7	-0.02% $\odot$	4.67% ✓
0.8	0.11% $\odot$	5.23% ✓
0.9	0.25% $\odot$	3.96% ✓
1	0.50% ✓	5.26% ✓
1.05	0.24% $\odot$	4.40% ✓
1.1	0.46% $\odot$	5.82% ✓
1.15	0.52% ✓	5.04% ✓
1.2	0.43% $\odot$	5.51% ✓

Table 2.7: Impact of coefficients of variation on the performance of the PDCS algorithm. Listed here are the averages of our simulation results. We varied the coefficient of variation of the more complex job type (the one requiring both service parts). The coefficient of variation of the other job type was determined such that the same variance-to-mean ratio was obtained for its negative binomial demand distribution. All other parameters took their base-case values. A check-mark indicates a statistically significant superiority at the 95% level and a cross indicates the opposite. A dot indicates that the difference is not statistically significant.

	I	J	L	H	means
1	3	2	(2,8,6)	(0.49,1.39,2.73)	(9,10)
2	4	3	(8,6,1,3)	(2.73,4.79,4.8,0.79)	(9,10,2)
3	5	4	(6,1,3,5,8)	(4.82,0.79,4.85,4.79,2.43)	(9,10,2,10)
4	5	4	(8,8,6,3,4)	(10,7,1,5,4)	(8,5,8,8)
5	6	5	(1,3,5,8,8,2)	(4.85,4.8,2,4.00,0.71,2.11)	(9,10,2,10,7)
6	6	15	(1,8,7,7,3,2)	(6,6,8,4,28,27)	(6,8,3,3,10,5,4,4,8,8,1,3,9,10,6)
7	6	15	(5,7,6,4,7,4)	(1,7,8,7,25,25)	(5,3,1,4,4,8,3,6,5,4,10,10,2,9,7)
8	6	15	(6,4,2,7,7,4)	(10,2,8,4,25,24)	(7,3,4,4,1,3,9,9,9,3,7,5,10,6,7)
9	6	15	(2,3,8,3,5,3)	(10,7,2,1,26,26)	(9,5,2,1,2,10,4,4,7,9,7,9,7,9,6)

Table 2.8: Listed here are the parameters randomly generated for each of the larger problems. I corresponds to the number of service parts involved and J corresponds to the number of job types involved. In each of these simulations, demands occur over 32 periods and the planning horizon extends to the $32 + \max_{i \in \mathcal{I}} L_i$-th period. The backordering costs are set to be equal to $5 \times \vec{h}' \times R$ where $\vec{h}$ is the vector of holding costs and R is the bill-of-materials matrix.

	MYBS - PDBS	MYBS - PDCS(*)
1	0.00% $\odot$	0.66% $\odot$
2	0.00% $\odot$	1.98%✓
3	0.00% $\odot$	-8.33% $\otimes$
4	0.00% $\odot$	2.81%✓
5	0.04% $\odot$	1.11%✓
6	0.00% $\odot$	2.21%✓
7	0.02% $\odot$	2.30%✓
8	0.01% $\odot$	3.56%✓
9	0.01% $\odot$	3.20%✓

Table 2.9: Listed here are the averages of our simulation results for each of the larger problems. A check-mark indicates a statistically significant superiority at the 95% level and a cross indicates the opposite. A dot indicates that the difference is not statistically significant.

CHAPTER 3

OPTIMALITY OF BASE-STOCK POLICIES UNDER AGE-DEPENDENT BACKORDER AND HOLDING COSTS

3.1 Chapter Abstract

We study inventory control problems where the cost of a backorder or the holding cost of a unit of inventory depends on how long the backorder or the unit of inventory has been in the system. It is expected that a standard dynamic programming formulation of the problem will require a high-dimensional state vector that keeps track of the backorders and inventory with different ages. In Huh et al. (2011), it is shown that a dynamic program with a scalar state variable can be set up. Subsequently, the optimality of base-stock policies is established. In this chapter, we present our independent development of a different dynamic program with a scalar state variable by using an alternative cost accounting mechanism. Using this alternate approach, we also show that base-stock policies are optimal. This dynamic program is suitable for computation and it allows us to compute the optimal base-stock levels. We numerically compare the optimal policy with two others: (1) a standard myopic policy where the one-period cost function minimized coincides with that appearing in the formulation of Huh et al. (2011), and (2) an alternate myopic policy based on the immediate cost function in our formulation. The alternate myopic policy performs well and is marginally better than the standard myopic policy.

3.2 Introduction and Literature Review

We consider a single-product inventory control problem where the cost of a backorder or the holding cost of a unit of inventory depends on how long the backorder or the unit of inventory has been in the system. We particularly focus on the case where the per-time period backorder or inventory holding cost increases, the longer the backorder or the unit of inventory has been in the system. Such an age-dependent backorder cost arises in contractual repair settings where customers impose service penalties for failing to meet promised repair times. On the other hand, service parts that suffer deterioration with age require some restoration or price adjustment when they are finally used. The cost of these adjustments can be viewed as an age-dependent holding cost.

As shown in Huh et al. (2011), the state space for such problems can be collapsed to a single state variable and base-stock policies are optimal. In other words, there is a base-stock level I_t^* for each time period t so that if the inventory position at the beginning of time period t is below I_t^*, then it is optimal to raise the inventory position to I_t^*. We establish the same result but using a different approach. A standard dynamic programming formulation of the problem is expected to use a vector state variable which keeps track of the backorders and inventory with different ages. Our proof of optimality for base-stock policies uses two key steps to address this difficulty. First, we use a cost accounting mechanism which computes the total expected holding and backorder cost that we incur due to each incremental unit of inventory from the time of purchase until this unit of inventory leaves the system. We charge this total expected cost when we purchase the unit of inventory. This cost accounting mechanism allows us to formulate a dynamic program with only a scalar state variable. Second, even under the new cost accounting scheme, the value functions in our dynamic program may not be convex. We then shift our value functions by adding appropriate functions to them to obtain a new dynamic program whose value functions turn out to be convex. By using the convexity of the shifted value functions, we prove the optimality of base-stock policies.

The results we give in this chapter have several useful theoretical and practical implications. By individually accounting for the total expected holding and backorder cost incurred due to each unit of inventory, we formulate the problem as a dynamic program with a scalar state variable. Our work looks at the state of the system and costs in an alternative fashion, which may be useful in other settings. Besides its theoretical appeal, optimality of base-stock policies is practically useful. One can characterize a base-stock policy simply by specifying a single scalar at each time period. Implementing such policies in practice is much easier than implementing those that depend on the state of the system in a complicated fashion. Our work is also practical with our dynamic program providing a computationally tractable tool to compute the optimal base-stock levels. We present a short numerical illustration at the end of the chapter. Our dynamic programming formulation naturally motivates a non-standard myopic policy, where we simply minimize the immediate expected cost component that is driven by our alternative cost accounting mechanism. We compare our optimal policy with this alternate myopic policy as well as a myopic policy motivated by the immediate cost function found in the formulation by Huh et al. (2011). The alternate myopic policy performs well and is marginally better than the standard myopic policy. The results in this chapter give a complete analysis for inventory control problems with age-dependent backorder and inventory holding costs, providing theoretical characterization of the optimal policy, building computational tools to compute the optimal policy and constructing a heuristic myopic policy that performs well.

In addition to Huh et al. (2011), our approach in this paper has particularly strong ties to three papers in the literature. Our alternative cost accounting scheme is inspired by Muharremoglu and Tsitsiklis (2008), where the authors analyze multiple-echelon inventory systems by keeping track of when each unit of inventory is matched up with a customer order. Levi et al. (2007) use a cost accounting mechanism that keeps track of the total expected holding and backorder cost incurred due to each incremental unit of inventory. Their focus is on building approximation algorithms for inventory control problems with general demand processes. Axsater (1990) uses a cost accounting mechanism similar to ours to evaluate the performance of base-stock policies in an inventory distribution system.

In the literature, each unit of inventory or backorder usually incurs the same cost per time period and the cost structure in our model is nontraditional in the sense that it considers age-dependent costs. In addition to the usual linear backorder cost which is per unit backorder per time period, Rosling (2002) mentions two forms of backorder costs, one charging a per unit backorder cost irrespective of the duration of the backorder and the other charging a per period backorder cost irrespective of the magnitude of the backorder. None of these cost structures captures our age-dependent backorder costs. Perishable inventory systems are relevant to our work as they also consider the age aspect of inventory. Earlier results are reviewed in Nahmias (1982) and Goyal and Giri (2001). The recent book by Nahmias (2011) covers a variety of perishable inventory models. The difference between perishable inventory models and our work is that inventory in our model always stays in the system until it is used to satisfy a demand, but it incurs a progressively larger holding cost. Base-stock policies turn out to be optimal in a variety of settings. Porteus (2002) and Zipkin (2000) are comprehensive references on inventory control models that admit base-stock policies as optimal policies.

The rest of the chapter is organized as follows. Section 3.3 describes our cost accounting mechanism and provides a dynamic program formulation of the inventory control problem with age-dependent costs. Section 3.4 establishes the optimality of base-stock policies. Section 3.5 discusses the standard myopic policy found in the formulation of Huh et al. (2011). Section 3.6 gives a numerical illustration and demonstrates the effectiveness of a myopic policy driven by our cost accounting mechanism.

3.3 Problem Formulation

We control the inventory of a product, where the per-time period cost of a backorder and the per-time period holding cost of a unit of inventory depend, respectively, on how long the backorder and the unit of inventory have been in the system. There are T time periods in the planning horizon. We use D_t to denote the integer-valued random demand in time period t. Demands in different time periods are independent. We receive replenishment orders after a lead time of L time periods. If a backorder has been in the system for i time periods, then we incur a cost of π^i per time period that we retain the backorder in the system. We assume that $\pi^1 \leq \pi^2 \leq \ldots$, that is, the per-time period cost of backorders is non-decreasing in the age of the backorder. Similarly, if a unit of inventory has been in the system for i time periods, then we incur a holding cost of h^i per time period that we retain the unit of inventory in the system. We assume that $h^1 \leq h^2 \leq \ldots$ so that the per-time period holding cost of the units is non-decreasing in the age of the inventory. The backorder and holding cost parameters are stationary and they depend only on how long a backorder or a unit of inventory has been in the system. Similarly, we assume that the purchasing cost is stationary and we set the unit purchasing cost to zero without loss of generality.

The following sequence of events take place in time period t. First, we observe the inventory position. Following standard terminology, the inventory position is given by the number of units on hand, plus the number of units on order, minus the number of backorders. Second, we place a replenishment order and we receive the replenishment order that was placed in time period $t - L$. Third, we observe the demand in time period t and satisfy the demand as much as possible by using the inventory on hand. If the backorders exceed the inventory on hand, then we give priority to satisfying the older backorders first since the unit cost of older backorders is larger. On the other hand, if the inventory on hand exceeds the backorders, then we give priority to using the older units of inventory first to satisfy the demand since the unit holding cost of older units is larger. We incur the inventory holding and backorder costs based on the ending inventory and backorders in time period t.

We are interested in finding a replenishment policy that minimizes the total expected holding and backorder cost over the planning horizon. The demand takes place over T time periods and we make the last inventory replenishment decision in time period $T + 1$. The last replenishment quantity is received in time period $T + L + 1$. Although there is no demand between time periods $T + 1$ and $T + L$, we can incur holding and backorder costs between these two time periods due to the replenishment decisions made at or before time period $T + 1$. Therefore, we choose to minimize the total expected cost over $T + L + 1$ time periods as the objective in our model. This is a standard way of addressing the end of horizon effects in inventory control models with replenishment lead times.

If we formulate the problem as a dynamic program by using a standard cost accounting mechanism that keeps track of the total expected holding and backorder cost incurred at each time period, it is expected that the state vector will keep track of the numbers of units on hand and backorders that have been in the system for different numbers of time periods. Such an approach will end up with a high-dimensional state vector. To get around this difficulty, we use a different cost accounting mechanism. In particular, each unit of inventory stays in the system for a certain duration of time and it is used to satisfy a unit of demand that arrives at a certain time period. Therefore, we can attribute an expected holding and backorder cost to each unit of inventory depending on how long the unit of inventory stays in the system before it is used to satisfy a unit of demand and how long the unit of demand in question stays in the system before it is satisfied. Furthermore, since the holding cost of a unit of inventory increases as the unit stays in the system longer and the cost of a backorder increases as the backorder stays in the system longer, it is optimal to serve the oldest backorder by using the oldest unit of inventory. Thus, we can use these observations to probabilistically characterize how long a unit of inventory stays in the system before it is used to satisfy a unit of demand and how long a unit of demand stays in the system before it is satisfied. We proceed to give the details of our cost accounting mechanism.

With the assumption that the purchasing cost is constant, if the inventory position at the beginning of time period t is negative, it is optimal to purchase at least an adequate amount of inventory in this time period to raise the inventory position to zero. This implies the optimal order-up-to level must be at least equal to zero. In formulating the problem, however, we give ourselves the option of not ordering up to at least zero. Although the optimal order-up-to level will never be negative when the purchasing cost is constant, this formulation allows us to more conveniently extend our work to a multi-echelon situation in the next chapter and to cases where the purchasing cost changes over time. Given the option to not order up to zero, we assume that a backorder continues to accumulate backorder charges at π^{L+1} per period beyond $L+1$ periods. Note that if the inventory position at the beginning of time period t is negative, this negative inventory position is due to the demand that arrived in time period $t-1$ as well as the outstanding backorders that we chose not to satisfy when we ordered at $t-1$. As far as charging the remaining per-period backorder cost is concerned, it is not necessary to distinguish among backorders $L+1$ periods old and up. We use I_t to denote the inventory position at the beginning of time period t before placing a replenishment order. In this case, considering the qth unit of inventory that we purchase in time period t, if we have $I_t + q \leq 0$, then the qth unit of inventory that we purchase in time period t is definitely used to satisfy a unit of demand that either arrived in time period $t-1$ or remained as a backorder following the purchase decision at $t-1$. On the other hand, if we have $I_t + q > 0$, then the qth unit of inventory that we purchase in time period t is used to satisfy a unit of demand that arrives in one of the time periods

$t, t+1, \ldots$. Which one of these time periods end up being the time period of consumption is unknown at the time of purchase.

We define the random variable $\theta_t(q, I_t)$ such that if the inventory position at the beginning of time period t is I_t, then the qth unit of inventory that we purchase at this time period is used to satisfy the demand that arrives in time period $\theta_t(q, I_t)$ if $I_t + q > 0$. In the case that $I_t + q \leq 0$, we set $\theta_t(q, I_t) = t - 1$ with probability 1 even though the demand satisfied by the qth unit could have arisen in a period before $t - 1$. The difference between using the qth unit to satisfy a demand from $t - 1$ and using it to satisfy a demand before $t - 1$ is that the demand from before $t - 1$ will have to stay in the system for more than $L + 1$ periods. Provided that we have already charged the linear backorder cost, π^{L+1} per period, beyond $L + 1$ periods associated with the demand from before $t - 1$, there is no further difference between the two demands and we can treat them both as if they are backorders from period $t - 1$. It is, therefore, appropriate mathematically to set $\theta_t(q, I_t) = t - 1$ when $I_t + q \leq 0$. When $I_t + q > 0$, we observe that the qth unit of inventory that we purchase in time period t is used to satisfy a unit of demand that arrives in time period τ if and only if τ is the first time period such that the cumulative demand over the time periods $t, t+1, \ldots, \tau$ exceeds $I_t + q$. Therefore, we can give a characterization of the random variable $\theta_t(q, I_t)$ in terms of the demand random variables as

$$\theta_t(q, I_t) = \begin{cases} \min\{\tau : D_t + D_{t+1} + \ldots + D_\tau \geq I_t + q\} & \text{if } I_t + q > 0 \\ t - 1 & \text{if } I_t + q \leq 0. \end{cases} \tag{3.1}$$

Throughout this chapter, we follow the convention that $D_{T+1} = D_{T+2} = \ldots = D_{T+L} = 0$ and $D_{T+L+1} = \infty$. In this case, we have no demand arriving between time periods $T + 1$ and $T + L$ and $\theta_t(q, I_t)$ is always well-defined, satisfying $\theta_t(q, I_t) \leq T + L + 1$ with probability 1. Also, we note that $\theta_t(q, I_t) = \theta_t(q + I_t, 0)$

by the definition of $\theta_t(q, I_t)$ in (3.1). This observation becomes useful in our proofs.

The qth unit of inventory purchased in time period t arrives in time period $t + L$. Furthermore, if the inventory position at the beginning of time period t is I_t, then this qth unit of inventory is used to satisfy a unit of demand that arrives in time period $\theta_t(q, I_t)$ (or a backorder remaining at the end of $\theta_t(q, I_t) = t - 1$). Thus, if we have $\theta_t(q, I_t) > t + L$, then the qth unit of inventory stays in the system for time periods $t+L, t+L+1, \ldots, \theta_t(q, I_t)-1$ before it is used to satisfy a unit of demand. So, it incurs a total holding cost of $h^1+h^2+\ldots+h^{\theta_t(q,I_t)-(t+L)}$. On the other hand, if we have $\theta_t(q, I_t) < t+L$, then the qth unit of inventory is used to satisfy a unit of demand that arrives at a time period before $t+L$ and this unit of demand stays in the system for time periods $\theta_t(q, I_t), \theta_t(q, I_t)+1, \ldots, t+L-1$. In this case, we can attribute a backorder cost of $\pi^1 + \pi^2 + \ldots + \pi^{(t+L)-\theta_t(q,I_t)}$ up to $L + 1$ periods to the qth unit of inventory that we purchase in time period t. Therefore, if the inventory position at the beginning of time period t is I_t, then the qth unit of inventory that we purchase at this time period incurs a total holding and backorder cost up to $L + 1$ periods of

$$\Gamma_t(\theta_t(q, I_t)) = \sum_{i=1}^{\theta_t(q,I_t)-(t+L)} h^i + \sum_{i=1}^{(t+L)-\theta_t(q,I_t)} \pi^i, \tag{3.2}$$

where we follow the convention that sums over empty index sets are zero. It is useful to see that $\Gamma_t(n) = \sum_{i=1}^{n-(t+L)} h^i + \sum_{i=1}^{(t+L)-n} \pi^i$ is a convex function in its argument n. In particular, we have

$$\Gamma_t(n) - \Gamma_t(n-1) = \begin{cases} -\pi^{(t+L)-n+1} & \text{if } n \leq t+L \\ h^{n-(t+L)} & \text{if } n > t+L \end{cases}$$

so that the convexity of $\Gamma_t(\cdot)$ follows by noting that $\ldots - \pi^2 \leq -\pi^1 \leq h^1 \leq h^2 \leq \ldots$. Another useful observation is the identity $\Gamma_t(n) = \Gamma_{t+1}(n+1)$, which follows from the definition of $\Gamma_t(n)$.

We let r_t denote the inventory position in time period t after placing the replenishment order but before observing the demand in this time period. We formulate the problem as a dynamic program by using I_t as the state variable and r_t as the decision variable. In this case, the optimal policy can be found by computing the value functions through the optimality equation

$$V_t(I_t) = \min_{r_t \geq I_t} \left\{ \sum_{q=1}^{r_t - I_t} \mathbb{E}\{\Gamma_t(\theta_t(q, I_t))\} + \pi^{L+1}[-r_t]^+ + \mathbb{E}\{V_{t+1}(r_t - D_t)\} \right\}, \quad (3.3)$$

where we use $[\cdot]^+ = \max\{\cdot, 0\}$. Without requiring r_t to be greater than $[I_t]^+$ which is the optimal thing to do when the purchasing cost is constant, we must charge ourselves π^{L+1} per unit of outstanding baackorder that we choose not to satisfy now. This is a per-period backorder cost beyond $L+1$ periods. Including this term allows us to treat the backorders which we carry from the current period to the next as well as the backorders which are due to the demand over t in the same manner when we arrive at $t+1$. Raising the inventory position to r_t means a purchase of $r_t - I_t$ units and the summation in (3.3) accounts for the total expected holding and backorder cost up to $L+1$ periods associated with each of these units. For the boundary condition of the optimality equation, we assume that if the inventory position at the end of the planning horizon is negative, then a last purchase is required to exactly satisfy the backorders so that

$$V_{T+1}(I_{T+1}) = (\pi^1 + \pi^2 + \ldots + \pi^{L+1})\,[-I_{T+1}]^+. \quad (3.4)$$

This terminal function ensures that all remaining backorders at $T+1$ are served in period $T+L+1$ by the inventory purchased in time period $T+1$.

Both the state variable I_t and the decision variable r_t appear in the upper bound of a summation in the optimality equation in (3.3). This makes it difficult to establish the convexity of the value functions. In the next section, we transform the value functions appropriately to obtain convex value functions, which ultimately allow us to establish the optimality of base-stock policies using this alternate cost accounting mechanism.

3.4 Optimality of Base-Stock Policies

Our objective in this section is to show that a base-stock policy is optimal for the optimality equation in (3.3). We begin by manipulating the immediate expected cost component $\sum_{q=1}^{r_t - I_t} \mathbb{E}\{\Gamma_t(\theta_t(q, I_t))\}$ in the optimality equation. Noting that $\theta_t(q, I_t) = \theta_t(q + I_t, 0)$, we have $\sum_{q=1}^{r_t - I_t} \mathbb{E}\{\Gamma_t(\theta_t(q, I_t))\} = \sum_{q=I_t+1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q - I_t, I_t))\} = \sum_{q=I_t+1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q, 0))\}$. Considering the cases $I_t > 0$ and $I_t \leq 0$ where $r_t \geq 0$ or $r_t < 0$ separately, we can write the latter summation as

$$\sum_{q=I_t+1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} = \begin{cases} \sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} - \sum_{q=1}^{I_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} & \text{if } 0 \leq I_t \leq r_t \\ \sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} + \sum_{q=I_t+1}^{0} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} & \text{if } I_t \leq 0 \leq r_t \\ \sum_{q=I_t+1}^{0} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} - \sum_{q=r_t+1}^{0} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} & \text{if } I_t \leq r_t < 0. \end{cases} \tag{3.5}$$

We have $\theta_t(q, 0) = t - 1$ whenever $q \leq 0$ by the definition of $\theta_t(q, I_t)$ in (3.1). Thus, the second summation in the second case on the right side above is given by $\sum_{q=I_t+1}^{0} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} = [-I_t]^+ \, \Gamma_t(t-1)$. Similarly, one can see that the expression for the last case is equal to $([-I_t]^+ - [-r_t]^+)\Gamma_t(t-1)$. Assembling these observations with (3.5), we can write the immediate expected cost component in (3.3) as

$$\sum_{q=1}^{r_t - I_t} \mathbb{E}\{\Gamma_t(\theta_t(q, I_t))\} = \sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} - \sum_{q=1}^{[I_t]^+} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\}$$
$$+ [-I_t]^+ \, \Gamma_t(t-1) - [-r_t]^+ \, \Gamma_t(t-1). \quad (3.6)$$

Noting that only the first and last terms on the right side above depends on the decision variable r_t in time period t, we can write the optimality equation in (3.3) as

$$V_t(I_t) = \min_{r_t \geq I_t} \Bigg\{ \sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} + [-r_t]^+ (\pi^{L+1} - \Gamma_t(t-1))$$
$$+ \mathbb{E}\{V_{t+1}(r_t - D_t)\} \Bigg\} - \sum_{q=1}^{[I_t]^+} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} + [-I_t]^+ \, \Gamma_t(t-1). \quad (3.7)$$

The appealing aspect of the optimality equation above is that the inventory position I_t does not appear in the objective function of the optimization problem in the curly braces. To obtain the desired result, we transform the value functions by defining

$$C_t(I_t) = V_t(I_t) + \sum_{q=1}^{[I_t]^+} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} - [-I_t]^+ \, \Gamma_t(t-1) \quad (3.8)$$

for all $t = 1, \ldots, T+1$. Using (3.8), we can write $V_{t+1}(r_t - D_t)$ in the optimality equation in (3.7) as

$$V_{t+1}(r_t - D_t) = C_{t+1}(r_t - D_t) - \sum_{q=1}^{[r_t - D_t]^+} \mathbb{E}\{\Gamma_{t+1}(\theta_{t+1}(q,0))\} + [D_t - r_t]^+ \, \Gamma_{t+1}(t).$$

Therefore, we can equivalently write the optimality equation in (3.7) as

$$C_t(I_t) = \min_{r_t \geq I_t} \Bigg\{ \sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} + [-r_t]^+ (\pi_{L+1} - \Gamma_t(t-1))$$
$$+ \mathbb{E}\{C_{t+1}(r_t - D_t)\} - \mathbb{E}\left\{ \sum_{q=1}^{[r_t - D_t]^+} \mathbb{E}\{\Gamma_{t+1}(\theta_{t+1}(q,0))\} \right\}$$
$$+ \mathbb{E}\{[D_t - r_t]^+\} \, \Gamma_{t+1}(t) \Bigg\}. \quad (3.9)$$

Throughout the rest of the chapter, we work with (3.9) and characterize the structure of the optimal policy by using this optimality equation. In the next lemma, we begin by giving the boundary condition of the optimality equation in (3.9).

Lemma 3.1. *The definition of $V_{T+1}(\cdot)$ in* (3.4) *and the definition of $C_t(\cdot)$ in* (3.8) *imply that $C_{T+1}(I_{T+1}) = 0$.*

Proof. By the definition of $\Gamma_t(\cdot)$ in (3.2), we have $\Gamma_{T+1}(T) = \pi^1 + \pi^2 + \ldots + \pi^{L+1}$. Thus, (3.4) and (3.8) imply that $C_{T+1}(I_{T+1}) = V_{T+1}(I_{T+1}) + \sum_{q=1}^{[I_{T+1}]^+} \mathbb{E}\{\Gamma_{T+1}(\theta_{T+1}(q,0))\} - [-I_{T+1}]^+ (\pi^1 + \pi^2 + \ldots + \pi^{L+1}) = \sum_{q=1}^{[I_{T+1}]^+} \mathbb{E}\{\Gamma_{T+1}(\theta_{T+1}(q,0))\}$. If $I_{T+1} \leq 0$, then the last summation is over an empty index set and we obtain the desired result. Throughout the rest of this proof, we assume that $I_{T+1} > 0$. The definition of $\theta_t(q, I_t)$ in (3.1), along with the convention that $D_{T+1} = D_{T+2} = \ldots = D_{T+L} = 0$ and $D_{T+L+1} = \infty$, implies that $\theta_{T+1}(q,0) = T + L + 1$ whenever $q > 0$. Furthermore, by the definition of $\Gamma_t(\cdot)$ in (3.2), we have $\Gamma_{T+1}(T + L + 1) = 0$. Thus, if $I_t > 0$, then we have $C_{T+1}(I_{T+1}) = \sum_{q=1}^{[I_{T+1}]^+} \mathbb{E}\{\Gamma_{T+1}(\theta_{T+1}(q,0))\} = \sum_{q=1}^{[I_{T+1}]^+} \Gamma_{T+1}(T + L + 1) = 0$. □

In the next lemma, we focus on the immediate expected cost component in the optimality equation in (3.9) and show that this cost component is convex in the decision variable r_t.

Lemma 3.2. *The immediate expected cost component*

$$\sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0))\} + [-r_t]^+(\pi_{L+1} - \Gamma_t(t-1)) \\ - \mathbb{E}\left\{ \sum_{q=1}^{[r_t - D_t]^+} \mathbb{E}\{\Gamma_{t+1}(\theta_{t+1}(q,0))\} \right\} + \mathbb{E}\{[D_t - r_t]^+\}\, \Gamma_{t+1}(t) \quad (3.10)$$

is a convex function of r_t over the domain $\mathbb{Z}$.

Proof. We show that the first difference of the immediate expected cost function with respect to r_t is non-decreasing in r_t. To facilitate our discussion, we condition on D_t and write the conditional version of the immediate expected cost function in (3.10) as

$$
\begin{aligned}
f_t(r_t, D_t) = &\sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q,0)) \mid D_t\} + [-r_t]^+(\pi^{L+1} - \Gamma_t(t-1)) \\
&- \sum_{q=1}^{[r_t - D_t]^+} \mathbb{E}\{\Gamma_{t+1}(\theta_{t+1}(q,0)) \mid D_t\} + [D_t - r_t]^+\, \Gamma_{t+1}(t). \qquad (3.11)
\end{aligned}
$$

The second conditional expectation could be replaced by an unconditional one since $\theta_{t+1}(q,0)$ is independent of D_t, but we keep the conditional expectation for notational uniformity. If we can show that $\mathbb{E}\{f_t(r_t, D_t) - f_t(r_t - 1, D_t)\}$ is non-decreasing in r_t for all $r_t = 1, 2, \ldots$, then the result follows.

First, we compute $f_t(r_t, D_t) - f_t(r_t - 1, D_t)$ when $r_t > D_t \geq 0$. Since $r_t > D_t \geq 0$, we have $\theta_t(r_t, 0) = \min\{\tau : D_t + D_{t+1} + \ldots + D_\tau \geq r_t\} = \min\{\tau : D_{t+1} + \ldots + D_\tau \geq r_t - D_t\} = \theta_{t+1}(r_t - D_t, 0)$ by the definition of $\theta_t(q, I_t)$ in (3.1). This fact could also be seen easily in Figure 3.1 which is a plot of cumulative demand beginning at time t. This figure is included at the end of the chapter.

Therefore, if $r_t > D_t \geq 0$, then the distributions of $\theta_t(r_t, 0)$ and $\theta_{t+1}(r_t - D_t, 0)$ conditional on D_t are identical. On the other hand, using the definition of $f_t(r_t, D_t)$ in (4.54), if $r_t > D_t \geq 0$, then the first difference $f_t(r_t, D_t) - f_t(r_t - 1, D_t)$ is given by

$$
\begin{aligned}
f_t(r_t, D_t) - f_t(r_t - 1, D_t) &= \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) \mid D_t\} - \mathbb{E}\{\Gamma_{t+1}(\theta_{t+1}(r_t - D_t, 0)) \mid D_t\} \\
&= \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) \mid D_t\} - \mathbb{E}\{\Gamma_{t+1}(\theta_t(r_t, 0)) \mid D_t\} \\
&= \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) - \Gamma_t(\theta_t(r_t, 0) - 1) \mid D_t\}, \qquad (3.12)
\end{aligned}
$$

where the second equality uses the fact that the distributions of $\theta_t(r_t, 0)$ and

$\theta_{t+1}(r_t - D_t, 0)$ conditional on D_t are identical as long as $r_t > D_t$ and the third equality uses the identity $\Gamma_t(n) = \Gamma_{t+1}(n+1)$ that follows from the definition of $\Gamma_t(\cdot)$ in (3.2).

Second, we compute $f_t(r_t, D_t) - f_t(r_t - 1, D_t)$ when $0 < r_t \leq D_t$ for $r_t = 1, 2, \ldots$. Since $D_t \geq r_t \geq 1$, the definition of $\theta_t(q, I_t)$ in (3.1) implies that $\theta_t(r_t, 0) = t$. Using the definition of $f_t(r_t, D_t)$ in (4.54), if $1 \leq r_t \leq D_t$, then the first difference $f_t(r_t, D_t) - f_t(r_t - 1, D_t)$ is given by

$$\begin{aligned} f_t(r_t, D_t) - f_t(r_t - 1, D_t) &= \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) \mid D_t\} - \Gamma_{t+1}(t) \\ &= \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) \mid D_t\} - \mathbb{E}\{\Gamma_{t+1}(\theta_t(r_t, 0)) \mid D_t\} \\ &= \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) - \Gamma_t(\theta_t(r_t, 0) - 1) \mid D_t\}, \end{aligned} \tag{3.13}$$

where the second equality uses the fact that if $D_t \geq r_t \geq 1$, then $\theta_t(r_t, 0) = t$. Combining (3.12) and (3.13), we obtain $f_t(r_t, D_t) - f_t(r_t - 1, D_t) = \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) - \Gamma_t(\theta_t(r_t, 0) - 1) \mid D_t\}$ whenever $r_t > 0$, in which case, taking expectations yields $\mathbb{E}\{f_t(r_t, D_t) - f_t(r_t - 1, D_t)\} = \mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) - \Gamma_t(\theta_t(r_t, 0) - 1)\}$.

The definition of $\theta_t(q, I_t)$ in (3.1) implies that $\theta_t(r_t, 0)$ is stochastically increasing in r_t in the sense that $\mathbb{P}\{\theta_t(r_t, 0) \geq \tau\}$ is non-decreasing in r_t for all τ. Since $\Gamma_t(\cdot)$ is convex, $\Gamma_t(n) - \Gamma_t(n-1)$ is non-decreasing in n. In this case, by Lemma 4.7.2 in Puterman (1994), it follows that $\mathbb{E}\{\Gamma_t(\theta_t(r_t, 0)) - \Gamma_t(\theta_t(r_t, 0) - 1)\}$ is non-decreasing in r_t whenever $r_t > 0$.

Lastly, we need to check the first difference $f_t(r_t) - f_t(r_t - 1)$ when $r_t \leq 0$. When $r_t \leq 0$, both of the summation terms appearing if the definition of f_t are equal to 0. Therefore,

$$\begin{aligned}f_t(r_t) - f_t(r_t - 1) &= [-r_t](\pi^{L+1} - \Gamma_t(t-1)) + \mathbb{E}\{[D_t - r_t]\}\Gamma_{t+1}(t) \\ &\qquad - [1 - r_t](\pi^{L+1} - \Gamma_t(t-1)) - \mathbb{E}\{[D_t - r_t + 1]\}\Gamma_{t+1}(t) \\ &= -\pi^{L+1} \end{aligned} \tag{3.14}$$

Note that $\Gamma_t(t-1)$ and $\Gamma_{t+1}(t)$ cancel out each other above. Because $-\pi^{L+1} = \Gamma_t(t) - \Gamma_t(t-1) \leq \mathbb{E}\{\Gamma_t(\theta_t(1,0)) - \Gamma_t(\theta_t(1,0)-1)\} = f_t(1) - f_t(0)$, we may now conclude that the function f_t is indeed convex for all r_t. $\square$

In the next theorem, we show the optimality of base-stock policies. Based on the convexity of the immediate expected cost component established in Lemma 3.2, the optimality of base-stock policies follows from a style of argument that is used often in the inventory control literature.

Theorem 3.3. *The value functions $\{C_t(\cdot) : t = 1, \ldots, T+1\}$ computed through the optimality equation in* (3.9) *are convex. Furthermore, there exists an optimal base-stock level $I_t^* \geq 0$ for each time period t so that it is optimal to raise the inventory position to I_t^* in time period t whenever the inventory position at the beginning of this time period is below I_t^* and it is optimal to purchase nothing in time period t whenever the inventory position at the beginning of this time period is above I_t^*.*

Proof. We show both statements in the theorem by using induction over the time periods. The convexity of $C_{T+1}(\cdot)$ follows by Lemma 3.1. The induction hypothesis is that $C_{t+1}(\cdot)$ is convex. Using $f_t(r_t)$ to denote the immediate expected cost component in (3.10), the optimality equation in (3.9) is

$$C_t(I_t) = \min_{r_t \geq I_t} \Big\{ f_t(r_t) + \mathbb{E}\{C_{t+1}(r_t - D_t)\} \Big\}. \tag{3.15}$$

By Lemma 3.2 and the induction assumption, the objective function of the optimization problem on the right side of (3.15) is convex over the domain $\mathbb{Z}$. If we let I_t^* be a minimizer of this objective function over the domain $\mathbb{Z}$, then an optimal solution to problem (3.15) is given by I_t^* if $I_t \leq I_t^*$ and is given by I_t if $I_t > I_t^*$. This shows that the optimal policy in time period t has the desired structure. In this case, we can write the optimal objective value of problem (3.15) as $C_t(I_t) = f_t(I_t^*) + \mathbb{E}\{C_{t+1}(I_t^* - D_t)\}$ if $I_t \leq I_t^*$ and $C_t(I_t) = f_t(I_t) + \mathbb{E}\{C_{t+1}(I_t - D_t)\}$ if $I_t > I_t^*$. Figure 3.2 plots $C_t(I_t)$ as a function of I_t. In particular, the dashed line plots $f_t(I_t) + \mathbb{E}\{C_{t+1}(I_t - D_t)\}$ over the domain $I_t \in \mathbb{Z}$ and the solid line plots $C_t(I_t)$. We observe that $C_t(I_t)$ is equal to a constant $f_t(I_t^*) + \mathbb{E}\{C_{t+1}(I_t^* - D_t)\}$ over the domain $I_t \in \{\ldots, I_t^* - 2, I_t^* - 1, I_t^*\}$ and $C_t(I_t)$ coincides with $f_t(I_t) + \mathbb{E}\{C_{t+1}(I_t - D_t)\}$ over the domain $I_t \in \{I_t^* + 1, I_t^* + 2, I_t^* + 3, \ldots\}$. Since I_t^* is a minimizer, we have $f_t(I_t) + \mathbb{E}\{C_{t+1}(I_t - D_t)\} \geq f_t(I_t^*) + \mathbb{E}\{C_{t+1}(I_t^* - D_t)\}$ for all $I_t \in \mathbb{Z}$. These facts are sufficient to establish that $C_t(\cdot)$ is convex and we obtain the desired result. $\square$

It is worthwhile to emphasize that working with the value functions $\{C_t(\cdot) : t = 1, \ldots, T+1\}$ instead of $\{V_t(\cdot) : t = 1, \ldots, T+1\}$ appears to be crucial in showing the optimality of base-stock policies using this alternate cost accounting mechanism. In particular, the function $\sum_{q=1}^{r_t} \mathbb{E}\{\Gamma_t(\theta_t(q, 0))\}$ does not necessarily have desirable convexity properties when viewed as a function of r_t. Therefore, the convexity of $C_t(\cdot)$ does not immediately yield convexity properties for $V_t(\cdot)$ through the definition in (3.8) and we are not able to provide any convexity results for the value functions $\{V_t(\cdot) : t = 1, \ldots, T+1\}$.

3.5 The Standard and Benchmark Approach

As mentioned in the introduction, the work presented in this chapter represents our independent development of a dynamic program with a scalar state variable using an alternative cost accounting mechanism. In the next section, we will compare our policy numerically with one that is driven by an immediate cost function derived using a standard approach in the inventory control literature. We describe its mathematical formulation in this section. Given the state vector of on-hand inventory and backorders with different ages, the one-period cost function obtained is a convex function in the order-up-to level. We then move on to the transformed cost function presented in Huh et al. (2011). Under the assumptions of their model, this transformed cost function turns out to be the same as the one obtained using the standard approach.

Defining M_{t-1} to be the age of the oldest piece of on-hand inventory at the end of period $t-1$, we let $(\tilde{o}_{t-1}^1, \tilde{o}_{t-1}^2, ..., \tilde{o}_{t-1}^{M_{t-1}})$ be the vector of on-hand inventory with ages 1 through M_{t-1} in period $t-1$ following the realization of D_{t-1} and the satisfaction of backorders. Similarly, let $(\tilde{b}_{t-1}^1, \tilde{b}_{t-1}^2, ..., \tilde{b}_{t-1}^{L+1})$ be the vector of outstanding backorders with ages 1 thourgh $L+1$ at this time. Note once again that because purchasing costs are unchanging from one period to the next, the inventory position should be raised to at least zero in every period. Because of this, the oldest backorder at the end of a period has an age that is at most $L+1$.

At the end of time period $t+L$, one can see that $\tilde{o}_{t+L}^1$, the on-hand inventory of age 1, is equal what remains of the quantity q_t ordered at t. In general, $\tilde{o}_{t+L}^j$ is equal to the remaining portion of q_{t+1-j} for $j = 1, ..., L+1$. For older inventory, one can also see that $\tilde{o}_{t+L}^j$ is what remains of $\tilde{o}_{t-1}^{j-L-1}$ for $j = L+2, ..., L+1+$

M_{t-1}. An analogous observation can be made regarding backorders. The oldest backorders at the end of $t+L$ correspond to any demands arising in period t which are still not satisfied by the end of $t+L$. Similarly, backorders with an age of L correspond to unsatisfied demands from period $t+1$ and so on such that backorders with an age of 1 correspond to unsatisfied demands from period $t+L$. Using our notation, $\tilde{b}^j_{t+L}$ is what remains of $D_{t+L+1-j}$.

Suppose that we start in period t, before receiving the order placed a lead time ago and before demand is realized, with parameters $(\tilde{o}^1_{t-1}, ..., \tilde{o}^{M_{t-1}}_{t-1})$ and $(q_{t-L}, ..., q_{t-1})$ where the latter represents the quantities ordered between $t-L$ and $t-1$. The following integer program with decision variables $\tilde{o}^j_{t+L},\ j = 1, ..., L+1+M_{t-1}$ and $\tilde{b}^j_{t+L},\ j = 1, ..., L+1$ can be set up to capture the one-period cost for $t+L$. The parameters I_t and r_t correspond respectively to the inventory position before ordering and the order-up-to level which satisfies the condition $r_t \geq [-I_t]^+$. Because the purchasing cost is constant, we will enforce that the order-up-to level be at least zero in this benchmark policy. Note that $r_t - I_t$ gives the quantity q_t ordered at t. Also, the notation $D^{t_2}_{t_1}$ is used to denote

$\sum_{s=t_1}^{t_2} D_s$.

$$\min \sum_{j=1}^{L+1+M_{t-1}} h^j \tilde{o}_{t+L}^j + \sum_{j=1}^{L+1} \pi^j \tilde{b}_{t+L}^j \tag{3.16}$$

$$\text{s.t.} \quad \tilde{o}_{t+L}^1 \le r_t - I_t, \tag{3.17}$$

$$\tilde{o}_{t+L}^j \le q_{t+1-j}, \quad j = 2, ..., L+1, \tag{3.18}$$

$$\tilde{o}_{t+L}^j \le \tilde{o}_{t-1}^{j-L-1}, \quad j = L+2, ..., L+1+M_{t-1}, \tag{3.19}$$

$$\sum_{j=1}^{L+1+M_{t-1}} \tilde{o}_{t+L}^j \ge r_t - D_t^{t+L}, \tag{3.20}$$

$$\tilde{b}_{t+L}^j \le D_{t+L+1-j}, \quad j = 1, ..., L+1, \tag{3.21}$$

$$\sum_{j=1}^{L+1} \tilde{b}_{t+L}^j \ge D_t^{t+L} - r_t, \tag{3.22}$$

$$\tilde{o}_{t+L}^j \ge 0, \quad j = 1, ..., L+1+M_{t-1},$$

$$\tilde{b}_{t+L}^j \ge 0, \quad j = 1, ..., L+1.$$

One can easily verify that this integer program is discretely convex in the parameter r_t (fixing all the other parameters). First, note that as long as the parameters are all integral, the linear-program relaxation will admit an integral solution which can be obtained using a greedy algorithm. This is because if $D_t^{t+L} - r_t > 0$, $\tilde{o}_{t+L}^j = 0$ necessarily and the solution to the linear program can be obtained by "binpacking" $D_t^{t+L} - r_t$ backorders as $\tilde{b}_{t+L}^j$ up to their upper limits $D_{t+L+1-j}$ for $j = 1, ..., L+1$. We start this process with $\tilde{b}_{t+L}^1$ because π^j is non-decreasing in j. Similarly, if $D_t^{t+L} - r_t < 0$, $\tilde{b}_{t+L}^j = 0$ for all j and we will "binpack" $r_t - D_t^{t+L}$ units of on-hand inventory as $\tilde{o}_{t+L}^j$ in the linear-program solution for $j = 1, ..., L+1+M_{t-1}$ up to their upper limit starting with $\tilde{o}_{t+L}^1$. Therefore, we will treat the optimization problem above as a linear program.

We start by fixing the values of all the parameters except for r_t and denoting the optimal value of the linear program as a function of r_t using $z\left(r_t \middle| I_t, \{\tilde{o}^k_{t-1}\}_{k=1}^{M_{t-1}},\ \{q_s\}_{s=t-L}^{t-1}, \{D_s\}_{s=t}^{t+L}\right)$. Suppose we are given r^1_t, r^2_t and $\lambda \in [0,1]$ such that $\lambda\, r^1_t + (1-\lambda)\, r^2_t$ is an integer. Suppose further that the linear program with parameter r^i_t attains its minimum at $\left(\{\tilde{o}^{k,i}_{t+L}\}_{k=1}^{L+1+M_{t-1}}, \{\tilde{b}^{k,i}_{t+L}\}_{k=1}^{L+1}\right)$. One can easily check that $\lambda\left(\{\tilde{o}^{k,1}_{t+L}\}_{k=1}^{L+1+M_{t-1}}, \{\tilde{b}^{k,1}_{t+L}\}_{k=1}^{L+1}\right) + (1-\lambda)\left(\{\tilde{o}^{k,2}_{t+L}\}_{k=1}^{L+1+M_{t-1}}, \{\tilde{b}^{k,2}_{t+L}\}_{k=1}^{L+1}\right)$ (not necessarily integral) is feasible for the linear program with parameter $\lambda\, r^1_t + (1-\lambda)\, r^2_t$. The objective value attained at this point is $\lambda z(r^1_t|...) + (1-\lambda) z(r^2_t|...)$. We know that this linear program admits an optimal integral solution whose objective value is at most $\lambda z(r^1_t|...) + (1-\lambda) z(r^2_t|...)$. Therefore, we can conclude that the function $z\left(r_t \middle| I_t, \{\tilde{o}^k_{t-1}\}_{k=1}^{M_{t-1}},\ \{q_s\}_{s=t-L}^{t-1}, \{D_s\}_{s=t}^{t+L}\right)$ is convex in r_t because $\lambda z(r^1_t|...) + (1-\lambda) z(r^2_t|...) \geq z(\lambda r^1_t + (1-\lambda) r^2_t|...)$. In the next section, we numerically test the optimal policy against a myopic policy driven by this standard one-period cost function.

We conclude this section by describing the approach used in Huh et al. (2011) where a generalized cost model involving a linear holding cost for on-hand inventory at the end of each period as well as three types of backorder costs is considered. There is a standard and linear backorder cost for each outstanding backorder at the end of a period. In addition, there is a single fixed charge incurred for every period that has backorders outstanding at the end. Lastly, there is also an age specific backorder cost like our model. For an exact comparison, we ignore the first two components of backorder costs considered in their work, and we assume $h^j = h$ in our setup as in Huh et al. (2011). With r_t and $\tilde{b}^j_{t+L}$ defined as before, the one lead-time look-ahead cost function formulated in Huh et al. (2011) is

$$\tilde{f}_t(r_t) = \mathbb{E}\left[h\,(r_t - D_t^{t+L})^+ + \sum_{j=1}^{L+1} \pi^j \tilde{b}^j_{t+L}\right]. \tag{3.23}$$

Note that this one-lead-time look-ahead cost function is exactly the same as the one described in the form of a linear program above. This means that the standard myopic policy we test in the next section simply makes use of this cost function in Huh et al. (2011). Instead of deducing the convexity of this function in the above form, Huh et al. (2011) make use of the relationship

$$\sum_{j=1}^{L+1} \pi^j \tilde{b}_{t+L}^j = \sum_{j=1}^{L+1} (\pi^j - \pi^{j-1}) \sum_{k=j}^{L+1} \tilde{b}_{t+L}^k \text{ where } \pi^0 = 0 \tag{3.24}$$

and

$$\sum_{k=j}^{L+1} \tilde{b}_{t+L}^k = \left(D_t^{t+L+1-j} - r_t\right)^+ . \tag{3.25}$$

The second equivalence can be seen easily by noting that $\sum_{k=j}^{L+1} \tilde{b}_{t+L}^k$ is the total number of backorders by the end of $t+L$ that have been around for at least j periods. These backorders correspond to demands arising no later than period $t+L+1-j$ which are still not satisfied by the end of $t+L$. Assuming that the per period backorder cost increases with age, i.e. $\pi^j - \pi^{j-1} \geq 0$ for $j > 1$, one can easily see that the one lead time look-ahead cost function in the form of

$$\tilde{f}_t(r_t) = \mathbb{E}\left[h\left(r_t - D_t^{t+L}\right)^+ + \sum_{j=1}^{L+1} (\pi^j - \pi^{j-1}) \left(D_t^{t+L+1-j} - r_t\right)^+ \right] \tag{3.26}$$

is a convex function in r_t.

The work of Huh et al. (2011) also deals with situations where π^j may not be increasing. Quasi-convexity can be established if the probability distributions of D_t satisfy certain assumptions. Under these conditions, one may still conclude that base-stock policies are optimal. These results are then extended to the case where fixed ordering costs are present and where orders need to be placed in batches.

3.6 Numerical Illustration

In this section, we demonstrate that computing the optimal policy through the optimality equation in (3.9) is computationally tractable. We also compare the myopic policy that is motivated by this optimality equation with that which is based on the more standard cost accounting mechanism described in the previous section. We experiment with the following three benchmark methods.

Optimal policy (OPT) This benchmark method is the optimal policy obtained by solving the optimality equation in (3.9). The most problematic aspect of solving the optimality equation in (3.9) is computing the distribution of the random variable $\theta_t(q, 0)$. To address this difficulty, we generate 10,000 sample paths of the demand random variables $D_1, \ldots, D_{T+L+1}$. For each sample path, we compute the realization of $\theta_t(q, 0)$ for all $t = 1, \ldots, T + L + 1$, $q = 0, \ldots, \bar{q}$ by using (3.1), where $\bar{q}$ is a generous upper bound on the inventory position in any time period. By using the 10,000 realizations for $\theta_t(q, 0)$, we obtain an empirical distribution for this random variable and use this empirical distribution to compute the expectation $\mathbb{E}\{\Gamma_t(\theta_t(q, 0))\}$. Once we have an empirical distribution for $\theta_t(q, 0)$, we can compute and store the cost component $\mathbb{E}\{\Gamma_t(\theta_t(q, 0))\}$ for all $t = 1, \ldots, t + L + 1$, $q = 0, \ldots, \bar{q}$ by using (3.2). The remaining expectations in the optimality equation in (3.9) are straightforward to compute since they involve only the random variable D_t. Therefore, the term $\mathbb{E}\left\{\sum_{q=1}^{[r_t - D_t]^+} \mathbb{E}\{\Gamma_{t+1}(\theta_{t+1}(q, 0))\}\right\}$ in (3.9) is computed first by obtaining a table of values for $\mathbb{E}\{\Gamma_t(\theta_t(q, 0))\}$ using the empircal distribution for $\theta_t(q, 0)$, and then by taking the outer expectation with respect to D_t.

Myopic policy based on the optimality equation (MYO) This benchmark method is a myopic policy that uses the immediate expected cost component in the optimality equation in (3.9). In particular, if we let $\tilde{I}_t$ be a minimizer of the immediate expected cost component in (3.10) over the domain $\mathbb{Z}$, then MYO uses $\tilde{I}_t$ as the base-stock level in time period t. We intuitively expect MYO to perform well since this benchmark method makes its purchasing decisions by accounting for the total expected holding and backorder cost that we can attribute to each unit of inventory until the unit of inventory is consumed by a unit of demand. In particular, if the unit of inventory is likely to stay in the system for a long period of time, then MYO takes this possibility into consideration.

Myopic policy based on standard cost accounting (STA) This one-period cost function associated with this policy is described in detail in the previous section. Again, this benchmark method is based on the observation that the replenishment order placed in time period t is received in time period $t+L$, which implies that the purchasing decision in time period t does not affect the costs incurred in time periods $t, t+1, \ldots, t+L-1$. In this case, STA makes the purchasing decision in time period t to minimize the expected holding and backorder cost incurred in time period $t+L$. This benchmark method is myopic in the sense that it does not consider the cost implications of the purchasing decision made in time period t on time periods beyond $t+L$. It is, again, a standard approach in the inventory control literature to construct a myopic policy by considering the impact of the current purchasing decision only on the expected cost incurred a lead time into the future. As described in the previous section, this is also the myopic policy one obtains using the formulation presented in Huh et al. (2011).

In our test problems, there are $T = 50$ time periods in the planning horizon. The holding cost is always equal to 1 so that $h^i = 1$. We experiment with three different forms for the backorder costs. In the first form, we have $\pi^i = 1 + 0.2\,(i-1)$. In the second form, we have $\pi^i = 1.2^i$. In the third form, we have $\pi^i = 2 + 2\,(i-1)$. A backorder never stays in the system for more than 20 time periods in our test problems. For $i = 1, \ldots, 20$, it is useful to observe that the backorder costs under the first form are smallest and the backorder costs under the third form are largest. We use $L = 5$ or $L = 10$ as the replenishment lead time. We work with two different demand profiles. The demand in time period t has always Poisson distribution with parameter λ_t, but the demand profiles differ in how we generate λ_t. In the first demand profile, which we refer to as profile "a," λ_t is sampled from the uniform distribution between 3 and 8. We sample a different value of λ_t for all $t = 1, \ldots, T$ once and fix them at their sampled values. In the second demand profile, which we refer to as profile "b," we have $\lambda_t = 5 + \sin(\pi t/25)$. The first profile corresponds to a case where the demand is relatively stationary, but the second profile provides a seasonal demand pattern.

Table 3.1 shows our numerical results. The first column in this table shows the characteristics of the test problems by using the triplet $\{\Pi, L, D\} \in \{1,2,3\} \times \{5,10\} \times \{a,b\}$, where Π is the form for the backorder costs with 1, 2 and 3 being the three forms for the backorder costs, L is the replenishment lead time, and D is the demand profile with a and b being the two demand profiles. The second, third and fourth columns show the total expected costs incurred by OPT, MYO and STA, respectively. We estimate these total expected costs by simulating the performance of the three benchmark methods over 500 sample paths. The fifth and sixth columns show the percent gap between the total expected costs incurred by OPT and the remaining two benchmark methods, respectively. This column also includes a check, "✓," to indicate a statistically significant gap at the 95% level.

In all of our test problems, computing the optimal policy takes a few seconds. Comparing the performances of OPT and MYO in Table 3.1, we observe that MYO performs quite well, providing total expected costs within 2.05% of the optimal. The larger performance gaps between OPT and MYO generally correspond to the test problems with longer replenishment lead times. From an intuitive perspective, this trend is expected since longer lead times render the system more inflexible in terms of catching up with unexpected backorders. There does not appear to be a consistent effect of the backorder costs or the stationarity of the demand on the performance of MYO. On the other hand, the performance gaps between OPT and STA tend to be somewhat larger than those between OPT and MYO. Although both MYO and STA are myopic greedy policies, MYO takes into account the possibility of an inventory unit staying in the system for a long time when it computes the total expected holding and backorder cost associated with a procured unit. This explains the marginal advantage of MYO over STA. However, as we can see, MYO and STA both perform reasonably well in all cases.

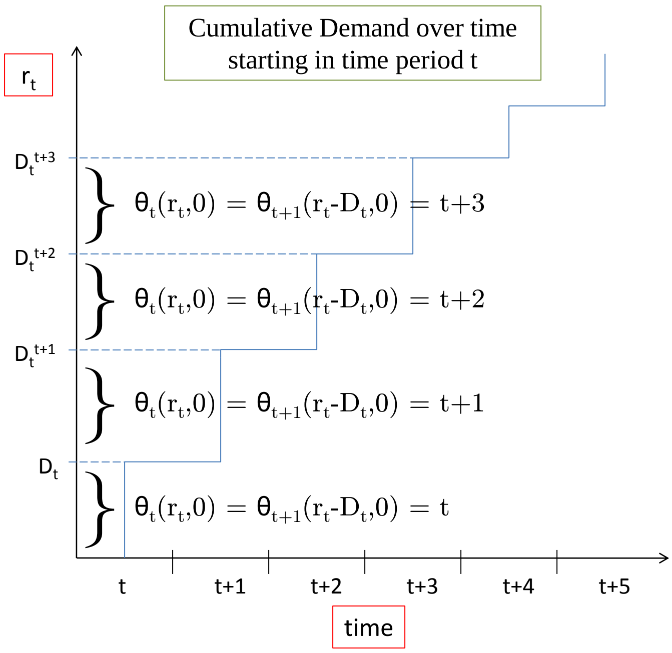

Figure 3.1: Cumulative demand starting at time t. It can be seen easily from this diagram that $\theta_t(r_t, 0) \overset{d}{=} \theta_{t+1}(r_t - D_t, 0)$ for $r_t > 0$ conditioning on D_t.

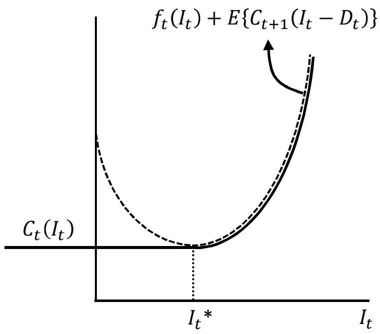

Figure 3.2: The functions $C_t(\cdot)$ and $f_t(\cdot) + \mathbb{E}\{C_{t+1}(\cdot - D_t)\}$.

Problem	Tot. Exp. Cost			OPT vs.	
(Π, L, D)	OPT	MYO	STA	MYO	STA
$(1, 5, a)$	237	239	241	0.94 ✓	1.72 ✓
$(1, 5, b)$	239	241	244	0.84 ✓	1.75 ✓
$(1, 10, a)$	371	376	379	1.35 ✓	2.13 ✓
$(1, 10, b)$	360	368	370	2.05 ✓	2.72 ✓
$(2, 5, a)$	271	273	274	0.70 ✓	1.15 ✓
$(2, 5, b)$	263	265	267	0.93 ✓	1.42 ✓
$(2, 10, a)$	411	415	420	1.00 ✓	2.10 ✓
$(2, 10, b)$	397	402	406	1.29 ✓	2.20 ✓
$(3, 5, a)$	367	369	372	0.50 ✓	1.29 ✓
$(3, 5, b)$	349	350	354	0.53 ✓	1.58 ✓
$(3, 10, a)$	578	583	594	0.93 ✓	2.74 ✓
$(3, 10, b)$	538	542	549	0.88 ✓	2.06 ✓

Table 3.1: Total expected costs incurred by OPT, MYO and STA.

CHAPTER 4

INVENTORY DISTRIBUTION PROBLEMS WITH AGING BACKORDERS

4.1 Chapter Abstract

We extend the results obtained in Chapter 3 to finite-horizon, two-echelon inventory distribution problems with aging backorders. The distribution system consists of a central warehouse which supplies a fixed number of downstream retailers by ordering its inventory from an external supplier with unlimited supply. Holding costs are charged linearly in the amount of on-hand inventory at the end of each period but the total backorder cost increases with the age of a demand in a convex manner. We formulate a dynamic program to obtain approximate order-up-to levels for the various locations in the distribution network using a cost mechanism that computes the expected backorder and holding cost associated with an ordered unit at the time of procurement. This dynamic program uses an immediate cost function which is a lower bound rather than an exact expression of the immediate cost function for the original system. Its state vector depends only on the echelon inventory positions of the various locations in the distribution network as well as the echelon net inventory and on-order inventory of the central warehouse. We further decompose this approximate dynamic program using the Clark-and-Scarf approach into multiple single-location inventory problems with aging backorders, each of which can then be solved using the method from Chapter 3. This decomposition provides a lower bound for expected cost in general. We present the results of our numerical study at the end of the chapter.

4.2 Introduction

In this chapter, we study a finite-horizon, two-echelon inventory distribution problem with aging backorders. This is an extension of the single-location inventory problem with aging backorders presented in the previous chapter. In an inventory distribution system, a central warehouse orders its inventory from an external supplier. The central warehouse in turn ships its inventory to a fixed number of downstream retailers. All the locations in the distribution network have procurement lead times. The following diagram illustrates a simple distribution system with three retailers and a central warehouse.

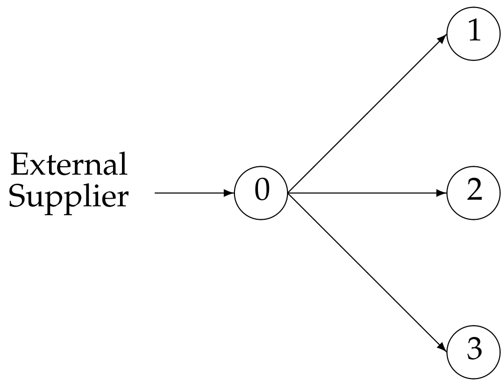

Figure 4.1: A distribution system with three retailers and a central warehouse.

In operating this system over a finite horizon, a linear holding cost is charged at the end of each period at each of the locations. Demands are realized only at the retailers and unfulfilled demands accumulate increasing per-period backorder costs. As mentioned in the previous chapter, increasing per-period backorder costs may arise in contractual repair settings where clients impose service penalties for failures to meet promised repair times.

For procurement, we propose using an order-up-to policy at the central warehouse. In particular, we compute a desired stock level for each time period such that if the echelon inventory position of the central warehouse (a notion we will define precisely in the next section) is below this desired level, we order a quantity that is equal to the difference between this desired level and the echelon inventory position. For the allocation of inventory among the retailers, we propose using a simple minimization problem which can be solved easily as outlined in detail in later sections.

In solving a single-location inventory problem with aging backorders, Huh et al. (2011) and we, in the previous chapter, independently propose two different cost accounting mechanisms, both of which collapse the state space in the exact dynamic program formulation into a single dimension. Huh et al. (2011) apply an interesting equivalence relationship to the traditional immediate cost function totalling the holding and backorder cost incurred at the end of a period to transform it into a sum of convex functions of the inventory position; the previous chapter uses a matching cost-accounting mechanism that computes the total expected holding and backorder cost associated with ordered units at the time of procurement. This mechanism turns out to depend only on the inventory position. Both of these approaches depend crucially for equivalence on the optimality of the first-come-first served allocation policy in a single-location inventory problem where the per-period per-unit backorder cost increases with age.

The solvability of the single-location problem with aging backorders using simply the inventory position as the state variable leads one to hope for an exact dynamic program formulation of the analogous distribution problem with a state vector which depends on the retailers only through their echelon inventory positions. However, because the first-come-first served allocation of inventory ordered by the central warehouse is not necessarily optimal, extra dimensions are needed in the state space and neither of the two approaches for the single-location problem can be applied directly. To get around this, we formulate a dynamic program which uses a modified version of the matching cost-accounting mechanism proposed in Chapter 3. This dynamic program provides a lower bound for the exact cost function and has a state vector that depends on the retailers only through their echelon inventory positions. We call this modified matching cost-accounting mechanism the **aggregate-matching cost-accounting mechanism** which we will describe in detail in later sections. After setting up this so called **lower-bound dynamic program**, we show that it can be be decomposed using the approach in Clark and Scarf (1960). We show that this decomposition generally provides a lower bound for the lower-bound dynamic program.

As mentioned earlier, single-location inventory systems with aging backorders are studied in Huh et al. (2011) and the previous chapter. We refer the reader to the introduction section of the previous chapter for other works in the literature that consider non-linear backorder costs. Provided that the first-come-first-served allocation of inventory remains optimal, the results presented in Huh et al. (2011) and the previous chapter can both be extended to a serial system consisting of a fixed number of locations arranged in tandem. This chapter contributes to the literature by proposing a computationally tractable algorithm which approximately solves a distribution inventory problem with aging backorders where the first-come-first-served allocation of ordered inventory may not be optimal. The lower bound obtained using the algorithm can be used to measure the performance of our operating policy. The algorithm we propose is based on the aggregate matching of ordered units with future demands. This matching approach was first introduced by other works in the literature included in the introduction section of Chapter 3.

The inventory distribution problem with non-aging inventory and backorders has been well studied in the literature. The holding and backorder costs incurred at the end of a period are typically assumed to be linear in the amount of on-hand inventory and backorders regardless of how long they have been in the system. A computationally tractable method for finding the exact solution is not known for the distribution system even in this case. Several representative works from this literature are particularly relevant to the results presented in this chapter. Clark and Scarf (1960) establish a decomposition technique for a general inventory system where each location is supplied by a unique upstream location. This decomposition technique separates the problem into multiple single-location problems with a single-dimensional state space each of which can be solved easily via backward recursion. This decomposition gives a lower bound for the value function of the dynamic program for the entire system. The work of Eppen and Schrage (1981) deals with a version of the problem where the costs are identical for the different retailers and the demands are independent, stationary and normal. The central warehouse is not allowed to hold any stock and the focus of their work is on getting an appropriate order-up-to quantity in each period for the central warehouse and on establishing an allocation policy that equates the stock-out probability at each of the retailers. The formulation presented in Federgruen and Zipkin (1984) resembles that in Clark and Scarf (1960) and the non-negativity constraints on the order quantities of the retailers are relaxed to obtain an operating policy driven by a lower-bound approximation. This result is further refined in Kunnumkal and Topaloglu (2008) where Lagrangian multipliers are introduced for the non-negative order constraints and an optimization procedure is performed over these multipliers to obtain a tighter lower bound.

The rest of this chapter is organized as follows. In the next section, we start by setting up the exact dynamic program. After that, we define some notation for our aggregate-matching cost-accounting mechanism. We then formulate the lower-bound dynamic program which has a state vector which depends on the retailers only through their echelon inventory positions. We go on to show that this dynamic program indeed provides a lower bound for the exact formulation. In section 4.4, we outline the Clark-and-Scarf approach which can be used to decompose the lower-bound dynamic program by defining the single-dimensional central warehouse and retailer problems. In section 4.5, we present the results of our numerical study.

4.3 The Exact and Lower-Bound Dynamic Program Formulations

We start by formulating the exact problem in this section. We then develop the aggregate-matching cost-accounting mechanism and the cost functions that will be used in an alternate dynamic program formulation. That this alternate dynamic program formulation gives a lower bound for the exact formulation will be discussed. We conclude this section by transforming the lower-bound dynamic program formulation so that we may apply the Clark-and-Scarf decomposition approach (Clark and Scarf (1960)) to approximately solve the problem.

4.3.1 The Exact Dynamic Program Formulation

We consider a two-echelon inventory distribution system with aging backorders over a finite horizon from period 1 to period T. A central warehouse replenishes its inventory by ordering from an external supplier. The central warehouse in turn supplies J downstream retailers. We label the central warehouse as location 0 and the retailers as location 1 through location J. Demands occur at the downstream retailers in every period up to period T. We use D_t^j to denote the random demand that occurs in period t at retailer j where $1 \leq j \leq J$. We use D_t^0 to denote the total demand at the J retailers in time period t where $1 \leq t \leq T$. In other words, $D_t^0 = \sum_{1 \leq j \leq J} D_t^j$ for $1 \leq t \leq T$.

All the locations in the distribution network have a constant procurement lead time. We use L_j to denote the constant procurement lead time for location j where $0 \leq j \leq J$. An order placed by location j in period t arrives at location j in period $t + L_j$. With no more demands after period T, we allow the central warehouse to place a final order on the external supplier at the beginning of $T+1$ so that all remaining backorders at each retailer j are satisfied no later than the end of period $T + L_0 + L_j + 1$. The retailers are allowed to order from the central warehouse until $T + L_0 + 1$ which is when the last order placed by the central warehouse at $T+1$ arrives. Before period $T + L_0 + 1$, the central warehouse may not have enough inventory to satisfy all the orders of the retailers.

We minimize the total cost incurred in the system up to the end of $T + L_0 + L_j + 1$ for retailers $j = 1$ through $j = J$ and up to the end of $T + L_0 + 1$ for the central warehouse. In each period, we need to decide how much inventory gets shipped from the external supplier to the central warehouse, and how much inventory gets shipped from the central warehouse to each of the retailers.

The following sequence of events takes place in period t. First, the quantity ordered a lead time ago arrives at each of the locations. Then, we decide on the order quantity for this period at each of the locations. Note specifically that while the order quantity placed on the external supplier is not limited, the central warehouse cannot ship more than it has. Therefore, there is a constraint on the total order placed on the central warehouse. After making these decisions, the unknown demand is realized at each of the retailers. Holding costs are charged at all the locations based on the amount of on-hand inventory at this point. Age-dependent backorder costs are also charged at all the retailers based on the number of remaining backorders of different ages.

At the end of each period until $T+L_0+L_j+1$ for the retailers and $T+L_0+1$ for the central warehouse, we let h_j be the per-period per-unit holding cost incurred at location j where $0 \leq j \leq J$. Additionally, we let π_i^j, $1 \leq j \leq J$, $i \geq 1$, be the per-period backorder cost incurred for a backorder of age i at location j. In other words, a demand at location j at time t which gets satisfied in period $t+s$ incurs a total backorder cost of $\sum_{i=1}^{s} \pi_i^j$. We use the notation Π_s^j to denote $\sum_{i=1}^{s} \pi_i^j$. We assume in our distribution problem that $\pi_i^j \leq \pi_{i+1}^j$ for all $i \geq 1$ where $1 \leq j \leq J$. That is, the per-period backorder cost increases with age. Note that this implies the optimality of the first-come-first-served allocation of inventory received by a retailer to outstanding orders. As for the unit cost of procurement, by assuming that the unit purchasing cost stays constant over time, we set it to be zero without any loss of generality.

We now define the state variables, the decision variables and some additional parameters which we will use to set up an exact dynamic program. We let $z_t = (z_t^0, z_t^1, ..., z_t^J)$ be the vector of echelon net inventory at the beginning of period t for locations 0 through J. The echelon net inventory of retailer j is defined to be the on-hand inventory minus the total number of backorders at retailer j. The echelon net inventory of the central warehouse is defined to be the on-hand inventory of the central warehouse plus the total on-order inventory to the retailers plus the total echelon net inventories of the retailers.

Furthermore, we let $Q_t = \{q_s^j\}_{t-L_j+1 \le s \le t-1, 0 \le j \le J}$ be the matrix of on-order inventory at the beginning of period t where q_s^j denotes the quantity ordered by location j at time s which is due to arrive at $s + L_j$. We also define for convenience the notion of echelon inventory position, I_t^j, for each of the retailers and the central warehouse. Echelon inventory position is echelon net inventory plus total on order for that echelon. Specifically, we define $I_t^j = z_t^j + \sum_{t-L_j+1 \le s \le t-1} q_s^j$.

Finally, we let $B_t = \{b_{jt}^s\}_{1 \le j \le J, 1 \le s \le M}$ be the matrix of unsatisfied backorders at the beginning of period t where b_{jt}^s denotes the number of backorders of age s remaining at retailer j. The parameter M signifies the maximum age and we can set it to be $T + L_0 + \max_{1 \le j \le J}\{L_j\}$ assuming that we start with no backorders at the beginning of period 1.

For decision variables, we let q_t^0 be the procurement quantity of the central warehouse for $1 \le t \le T + 1$ and we let q_t^j be the shipment quantities to retailer j for $1 \le T + L_0 + 1$. We require $q_t^j \ge 0$ for all j and t. We let r_t^j be the order-up-to echelon inventory position at j in period t such that $q_t^j = r_t^j - I_t^j$. The decision variables r_t^j where $0 \le j \le J$ allow us to characterize the decisions in period t in an alternate way.

To compute the on-hand inventory at the central warehouse at the end of period t, we subtract the total echelon inventory positions of the retailers from the echelon net inventory of the central warehouse. While the decision variables r_t^j where $1 \leq j \leq J$ have an immediate impact on the on-hand inventory at the central warehouse at the end of period t, the decision variable r_t^0 does not affect the on-hand inventory at the central warehouse until period $t + L_0$. If we look at the sum of all expected holding cost incurred at the central warehouse between period t and period $t + L_0$, we see that it is equal to

$$\sum_{s=t}^{t+L_0-1} h_0 \mathbb{E}\left(z_t^0 + \sum_{u=t-L_0+1}^{s-L_0} q_s^0 - \sum_{u=t}^{s-1} D_u^0 - \sum_{1\leq j\leq J} r_s^j\right) \\ + h_0 \mathbb{E}\left(r_t^0 - \sum_{u=t}^{t+L_0-1} D_u^0 - \sum_{1\leq j\leq J} r_{t+L_0}^j\right). \quad (4.1)$$

As for the holding cost incurred at the retailers, we see that the decision variables r_t^j where $1 \leq j \leq J$ affect the on-hand inventory of the retailers only through the net inventory, $r_t^j - \sum_{s=t}^{t+L_j} D_s^j$, at the end of period $t + L_j$. Combining the terms that depend on the decisions to be made in the current period, we get the following holding cost component of the immediate cost function to be used in the exact formulation:

$$-\sum_{1\leq j\leq J} h_0 r_t^j + h_0 \mathbb{E}\left(r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0\right) + \sum_{1\leq j\leq J} \mathbb{E} h_j (r_t^j - \sum_{s=t}^{t+L_j} D_s^j)^+. \quad (4.2)$$

As for the backorder costs incurred at the retailers, once we arrive at period t, we no longer have control over the charges incurred between the end of period t and the end of $t + L_j - 1$ at retailer j. The decision variables r_j^t affects the backorder costs incurred only through the backorders we have at the end of $t + L_j$, $b_{j,t+L_j+1}^s$ where $1 \leq s \leq M$. Because the per-period backorder cost increases with age, the oldest backorders at a retailer always get fulfilled first and we can characterize the expected backorder costs incurred at the end of $t + L_j$ at the retailers via $\sum_{1\le j\le J}\sum_{s=1}^{M}\pi_s^j b_{j,t+L_j+1}^s$ where $b_{j,t+L_j+1}^s$ is equal to the optimal solution in:

$$\min \sum_{1\le j\le J}\sum_{s=1}^{M}\pi_s^j b_{j,t+L_j+1}^s \tag{4.3}$$
$$\begin{aligned}
\text{s.t.}\quad & b_{j,t+L_j+1}^s \le D_{t+L_j+1-s}, \quad s=1,...,L_j+1, \quad j=1,...,J,\\
& b_{j,t+L_j+1}^s \le b_{jt}^{s-L_j-1}, \quad s=L_j+2,...,M, \quad j=1,...,J,\\
& \sum_{s=1}^{M} b_{j,t+L_j+1}^s \ge \sum_{u=t}^{t+L_j} D_u^j - r_t^j, \quad j=1,...,J,\\
& b_{j,t+L_j+1}^s \ge 0, \quad s=1,...,M, \quad j=1,...,J.
\end{aligned}$$

This mathematical program is easy to solve. If $\sum_{u=t}^{t+L_j} D_u^j - r_t^j \ge 0$, there will be backorders at the end of $t+L_j$ at retailer j and they must be the ones that have been backordered for the least amount of time.

We are now ready to formulate the **exact distribution problem with aging backorders**. For period $1\le t\le T+L_0+1$:

Exact Distribution Problem with Aging Backorders:

$$V_t^{\text{Exact}}(z_t,B_t,Q_t) = \min_{\substack{r_t\ge I_t,\\ \sum_{1\le j\le J} r_t^j \le z_t^0}} \left\{ -\sum_{1\le j\le J} h_0 r_t^j + \mathbb{1}_{\{t\le T+1\}} h_0 \mathbb{E}\left(r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0\right)\right.$$
$$\left. + \sum_{1\le j\le J}\mathbb{E}h_j(r_t^j - \sum_{s=t}^{t+L_j} D_s^j)^+ + \mathbb{E}\sum_{1\le j\le J}\sum_{s=1}^{M}\pi_s^j b_{j,t+L_j+1}^s + V_{t+1}^{\text{Exact}}(z_{t+1},B_{t+1},Q_{t+1})\right\} \tag{4.4}$$

where the backorders, b_{jt}^s, are goverened by (4.3). Note that the constraint $r_t \ge I_t$ ensures that we are always ordering non-negative quantities. The constraint $\sum_{1\le j\le J} r_t^j \le z_t^0$ ensures that we do not ship more than what we have at the central warehouse to the retailers. Also, with $T+1$ being the last period in which the central warehouse can place an order, we will force r_{T+1}^0 to be equal to $[-I^0_{T+1}]^+$ and we will force r^0_t to be equal to I^0_t for all $t > T+1$. The indicator function $\mathbb{1}_{\{t \le T+1\}}$ is included so that we are only minimizing the holding cost at the central warehouse up to the end of $T + L_0 + 1$. The boundary condition for this dynamic program is $V_{T+L_0+2} = 0$.

In order to apply the Clark-and-Scarf decomposition, we must be able to formulate a dynamic program whose state space depends on the retailers only through their echelon inventory positions. This means that the state vector cannot distinguish among backorders of different ages. Formulation (4.4) does not have this property. In the next subsection, we outline a modified version of the matching cost-accounting mechanism we used in Chapter 3 which will allow us to formulate a dynamic program that has this property.

4.3.2 The Aggregate-Matching Cost-Accounting Mechanism

We will apply the exact-matching cost-accounting mechanism used in Chapter 3 for the single-location problem with aging backorders at the level of the retailers in the distribution problem. As discussed in Chapter 3, this exact-matching cost-accounting mechanism allows us to collapse the state space into the single-dimensional inventory position. It assumes that beyond $L_j + 1$ periods, the per-period per-backorder cost stays constant at $\pi^j_{L_j+1}$ at retailer j. In the single-location problem, this does not cause any problem because no backorder should exist for larger than the single-location replenishment time plus one, $L_j + 1$. In the distribution system, that is no longer the case because of the additional lead time, L_0, required to replenish the central warehouse. Consider the retailer in isolation, no problems are caused if we assume that π^j_s is constant for $s \ge L_j + 1$. However, if π^j_s continues to increase beyond $i = L_j + 1$, there are some residual non-linear backorder costs that should be charged for backorders that stay for more than $L_j + 1$ periods at retailer j. We consider the inclusion of these residual costs at the level of the central warehouse.

In particular, suppose a unit of inventory is ordered at time t by the central warehouse to bring the echelon inventory position up to a certain level. This level of echelon inventory position is equal to the total excess inventory we have in the entire distribution system after the unit is ordered. If the aggregate demand at the retailers starting at t clears this excess inventory before the unit arrives at the central warehouse at $t + L_0$, we know for certain that an outstanding backorder at some retailer j at the time the excess gets cleared will have to wait until at least period $t + L_0 + L_j$ to be fulfilled. We call the time at which this clearance occurs the **aggregate matching time**.

The reason we cannot charge the exact expected residual non-linear backorder cost is that the first-come-first-served allocation of inventory ordered by the central warehouse is not necessarily optimal. We do not know which backorder among all the retailers at the aggregate matching time should eventually be fulfilled by this ordered unit. In fact, it could be the case that this ordered unit would eventually be matched up with a demand that has not even be realized yet. Its ultimate matching will likely depend on how demands unfold over the future periods. Probabilistically, we can describe easily when the aggregate matching occurs. Consequently, we will develop a lower-bound cost-accounting mechanism based on the aggregate matching time. We will call this cost-accounting mechanism for the distribution network as a whole the **aggregate-matching cost-accounting mechanism**.

We now define the stopping-time random variables to use in the aggregate-matching cost-accounting mechanism. For locations 0 through J, we define the random variable $\theta_t^j(q_j, I_t^j)$ as

$$\theta_t^j(q_j, I_t^j) = \begin{cases} \min\{\tau : D_t^j + D_{t+1}^j + \ldots + D_\tau^j \geq I_t^j + q_j\} & \text{if } I_t^j + q_j > 0, \\ t-1 & \text{if } I_t^j + q_j \leq 0. \end{cases} \tag{4.5}$$

For retailers 1 through J, $\theta_t^j(q_j, I_t^j)$ may be interpreted as the arrival time of the demand which gets matched with the q_j-unit ordered by retailer j at time t assuming that the starting echelon inventory position is I_t^j after receiving the order placed a lead time ago. This is completely analogous to the analysis presented in Chapter 3. We refer to θ_t^j as the matching time at retailer j. Note that if $I_t^j + q_j \leq 0$, we set $\theta_t^j(q_j, I_t^j)$ to be equal to $t-1$ even though the matching demand could have arrived earlier than $t-1$. Additional backorder costs incurred beyond $L_j + 1$ periods by demands arriving before period $t-1$ are charged in a different manner as we will soon describe.

For location 0, the central warehouse, the q_0-th ordered unit brings the echelon inventory position to $I_t^0 + q_0$ and $\theta_t^0(q_0, I_t^0)$ should be interpreted as the time at which the excess $I_t^0 + q_0$ is cleared by aggregate retailer demands. This is the aggregate matching time. When $I_t^0 + q_0 < 0$ and $\theta_t^0(q_0, I_t^0) = t-1$, there already exists a shortage in the distribution network and there is not enough inventory in the system even to cover all the existing backorders. In our decision problem, we will enforce the constraint that the quantity ordered by the central warehouse at time t is at least equal to $[-I_t^0]^+$. This can be done because there is no limit on how much can be ordered from the external supplier.

We make the assumption that demands occur up to period T and that total cost is incurred in the system up to the end of $T + L_0 + L_j + 1$ for the retailers and up to the end of $T + L_0 + 1$ for the central warehouse. We use the conventions

$D^j_{T+1} = D^j_{T+2} = ... = D^j_{T+L_0+L_j} = 0$ and $D^j_{T+L_0+L_j+1} = \infty$ for $0 \leq j \leq J$ to ensure that $\theta^j_t(q_j, I^j_t)$ is well defined for all j. (Note that for $j = 0$, we have the convention $D^0_{T+1} = D^0_{T+2} = ... = D^0_{T+2L_0} = 0$ and $D^0_{T+2L_0+1} = \infty$. Also, using these conventions, $\theta^j_t(q_j, 0) = T + L_0 + L_j + 1$ if $q_j > 0$ for $T < t \leq T + L_0 + L_j + 1$ and $0 \leq j \leq J$.)

We now define the cost functions used to formulate the alternate dynamic program whose state space depends only on the retailers through their echelon inventory positions. For the ordering decisions made by retailers 1 through J at time t, we define the following cost function to capture the total holding and backorder cost up to $L_j + 1$ periods associated with the q_j-th unit of inventory ordered by retailer j at time t when the echelon inventory position is I^j_t:

$$\Gamma^j_t(\theta^j_t(q_j, I^j_t)) = h_j \left(\theta^j_t(q_j, I^j_t) - (t + L_j)\right)^+ + \sum_{i=1}^{(t+L_j)-\theta^j_t(q_j, I_t)} \pi^j_i. \tag{4.6}$$

With the matching demand arriving at $\theta^j_t(q_j, I^j_t)$ and the procured unit arriving at the retailer at $t + L_j$, the procured unit will stay as on-hand inventory for $(\theta^j_t(q_j, I^j_t) - (t+L_j))^+$ periods and the matching demand will exist as a backorder for $(t + L_j - \theta^j_t(q_j, I^j_t))^+$ periods at the retailer. We use the convention that a sum over an empty index set is equal to 0. Note that the function $\Gamma^j_t(\cdot)$ is convex in its argument because

$$\Gamma^j_t(n) - \Gamma^j_t(n-1) = \begin{cases} -\pi^j_{(t+L_j)-n+1} & \text{if } n \leq t + L_j \\ h_j & \text{if } n > t + L_j \end{cases}$$

and π^j_i is assumed to be non-decreasing in i.

Define r^j_t to be the order-up-to echelon inventory position at retailer j for period t. Whenever we cannot set r^t_j to be larger than or equal to 0, $(-r^t_j)^+$ backorders remain unmatched with any units ordered by the retailer. These backorders will have to wait in the system for more than $L_j + 1$ periods. The backorder cost up to $L_j + 1$ periods for these backorders will be charged when the corresponding units are ordered in later periods. For the time being, we add the cost $\pi^j_{L_j+1}(-r^t_j)^+$ to capture a portion of the per-period backorder cost beyond $L_j + 1$ periods for these backorders. This may be only a portion of the per-period per-backorder cost beyond $L_j + 1$ periods because the per-period backorder cost incurred in the i-th period where $i > L_j + 1$ is equal to π^j_i which may be strictly larger than $\pi^j_{L_j+1}$. (A lower bound for the residual portion will be incorporated into a cost function which depends on the procurement decision at the central warehouse discussed later.) Summing it all up, we add to the time-t immediate cost function of our lower-bound dynamic program the following:

$$\sum_{1\le j\le J}\left(\sum_{q_j=1}^{r^j_t - I^j_t} \mathbb{E}\{\Gamma^j_t(\theta^j_t(q_j, I^j_t))\} + \pi^j_{L_j+1}[-r^j_t]^+.\right) \tag{4.7}$$

Note that the expression above only depends on the decision variables r^t_j and the state variables I^j_t for $1 \le j \le J$.

For the residual portion of the backorder cost beyond $L_j + 1$ periods which have not already been captured by the expression above, we define the following cost function

$$\Psi^j_t(\theta^0_t(q_0, I^0_t)) \quad = \quad \sum_{i=L_j+2}^{t+L_0+L_j-\theta^0_t(q_0,I^0_t)} \left[\pi^j_i - \pi^j_{L_j+1}\right]. \tag{4.8}$$

As discussed earlier, the aggregate matching time $\theta^0_t(q_0, I^0_t)$ should be interpreted as the time at which the excess inventory $I^0_t + q_0$ gets cleared by aggregate retailer demands. The function $\Psi^j_t(\theta^0_t(q_0, I^0_t))$ captures the exact residual backorder cost that should be charged if (and only if) the q_0-th unit ordered by the central warehouse will be matched with a demand at retailer j that arises in period $\theta^0_t(q_0, I^0_t)$.

A graphical illustration of this function is given in Figure (4.2). Note that if

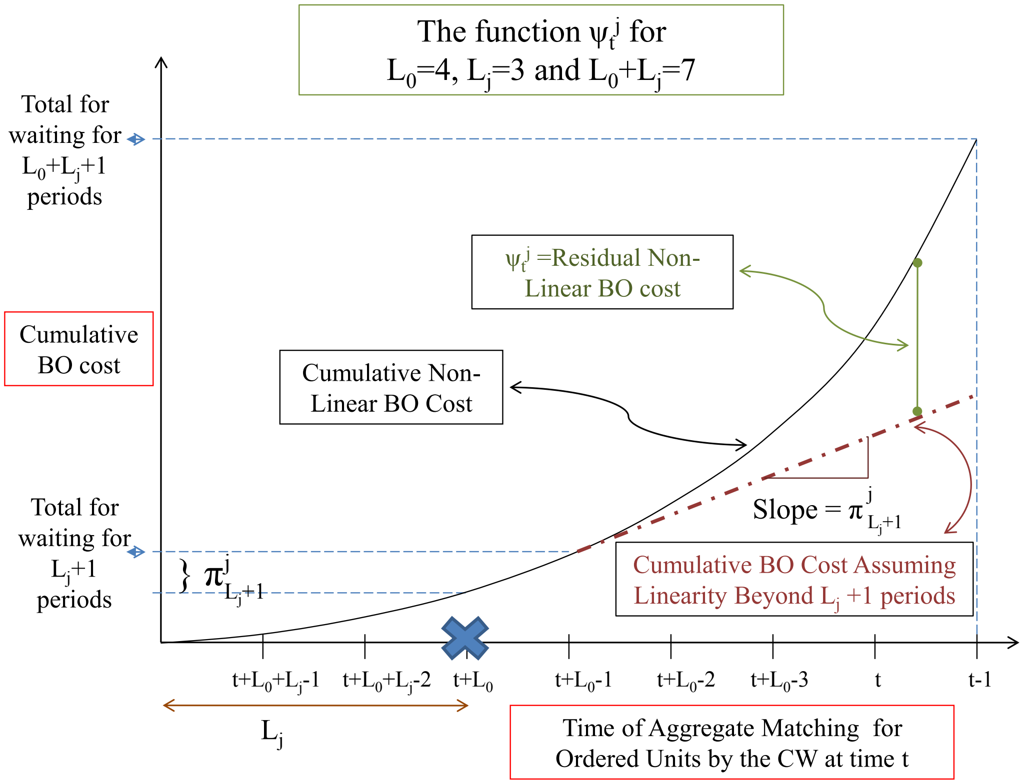

Figure 4.2: A plot showing $\Psi_t^j(\cdot)$ for $L_0 = 4$, $L_j = 3$, and $L_0 + L_j = 7$. The cross on the time axis indicates when orders placed by the central warehouse at t will arrive at the central warehouse.

π_i^j stays constant at $\pi^j_{L_j+1}$ for $i \geq L_j + 1$, the function $\Psi_t^j(\cdot)$ is equal to 0. The function Ψ_t^j is defined to capture the total per-period backorder cost in excess of $\pi^j_{L_j+1}$ beyond $L_j + 1$ periods. With

$$\Psi_t^j(n) - \Psi_t^j(n-1) = \begin{cases} \pi^j_{L_j+1} - \pi^j_{(t+L_0+L_j)-n+1} & \text{if } n < t + L_0, \\ 0 & \text{if } n \geq t + L_j, \end{cases}$$

we see that the function $\Psi_t^j(\cdot)$ is convex.

Whenever the aggregate matching time, θ_t^0, of a unit ordered at t is less than $t+L_0-1$, we know that a backorder at time θ_t^0 at some retailer j will have to wait until at least period $t+L_0+L_j$ to be fulfilled. Without knowing which retailer and which demand this unit will get matched with, we define the **aggregate residual backorder cost function**, Ψ_t^C, as the convex minorant of the individual retailer residual backorder cost functions

$$\Psi_t^C(\theta_t^0(q_0, I_t^0)) \stackrel{\text{def}}{=} \text{conv}\ \min_{1\leq j\leq J}\{\Psi_t^j(\theta_t^0(q_0, I_t^0))\} \tag{4.9}$$

which we will charge as a lower bound for the exact residual non-linear backorder cost incurred for demands that have to wait for more than L_j+1 periods at the retailers.

This subsection outlined the aggregate-matching cost-accounting mechanism as well as the special cost functions which we will use to capture the holding and backorder costs incurred at the retailers in the alternate dynamic program formulation. The next subsection states this alternate dynamic program formulation gives a lower bound for the exact formulation.

4.3.3 The Lower-Bound Dynamic Program Formulation

Combining the cost expressions (4.7) and (4.9) from the previous subsection with expression (4.2) which exactly captures the holding cost incurred at the central warehouse, we now define the lower-bound dynamic program formulation as follows:

Lower-Bound Formulation for the Dist. Problem with Aging Backorders:

$$\begin{aligned} V_t^{\mathrm{LB}}(I_t, z_t^0, Q_t^0) \quad = \quad & \min_{r_t \geq I_t} \left\{ \sum_{1\leq j\leq J} \left(\sum_{q_j=1}^{r_t^j - I_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, I_t^j))\} + \pi_{L_j+1}^j[-r_t^j]^+ \right) \right. \\ & + \sum_{q_0=1}^{r_t^0 - I_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, I_t^0))\} + \mathbb{1}_{\{t\leq T+1\}} h_0 \mathbb{E}\left[r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 \right] \\ & - \sum_{1\leq j\leq J} h_0 r_t^j + \mathbb{E}\{V_{t+1}^{\mathrm{LB}}(r_t - D_t, z_{t+1}^0, Q_{t+1}^0)\} \\ & \left. \text{such that } r_t^0 \geq [I_t^0]^+ \text{and} \sum_{1\leq j\leq J} r_t^j \leq z_t^0 \right\}. \end{aligned} \tag{4.10}$$

Here, I_t corresponds to the vector of echelon inventory positions. Also, z_t^0 and Q_t^0 correspond to the echelon net inventory at the central warehouse and the vector of on-order inventory to the central warehouse respectively. Note that $z_{t+1}^0 = z_t^0 + q_{t+1-L_0}^0 - D_t^0$ where $q_{t+1-L_0}^0$ is the quantity ordered by the central warehouse at $t+1-L_0$ which is part of the vector Q_t^0. The vector Q_{t+1}^0 is also easily obtained from Q_t^0 by removing the quantity $q_{t+1-L_0}^0$ and appending $r_t^0 - I_t^0$, which is the quantity ordered by the central warehouse at time t. Observe that the state space for the lower-bound dynamic program is much smaller than that for the exact dynamic program given in (4.4).

As in the exact formulation, we allow the central warehouse to place its last order at time $T+1$ and we force this order quantity to be equal to $[-I_{T+1}^0]^+$. We force all subsequent order quantities of the central warehouse to be zero. The indicator function $\mathbb{1}_{\{t\leq T+1\}}$ is included so that we are only minimizing the holding cost at the central warehouse up to the end of $T + L_0 + 1$. Because we are minimizing the cost incurred at retailer j up to the end of $T+L_0+L_j+1$ which is when the last potential backorder gets fulfilled, the appropriate terminal condition for this lower-bound dynamic program is $V_{T+L_0+1}^{\mathrm{LB}} = \sum_{1\leq j\leq J} \Pi_{L_j+1}^j[-I_{T+L_0+1}^j]^+$.

In order to reduce the dimension of the state space of the exact formulation, we used the aggregate-matching cost-accounting mechanism and used a cost-function which is only a lower bound of the exact cost function. We state the lower-bound result formally with

Lemma 4.1. *$V_t^{LB}(I_t, z_t^0, Q_t^0)$ as described by (4.10) and $V_t^{Exact}(z_t, B_t, Q_t)$ as described by (4.4) satisfy $V_t^{LB}(I_t, z_t^0, Q_t^0) \leq V_t^{Exact}(z_t, B_t, Q_t)$. Here $z_t = (z_t^0, z_t^1, ..., z_t^J)$ is the vector of echelon net inventory and $I_t = (I_t^0, I_t^1, ..., I_t^J)$ is the vector of echelon inventory positions. Also, $Q_t = \{q_s^j\}_{t-L_j+1 \leq s \leq t-1, 0 \leq j \leq J}$ is the matrix of on-order inventory where q_s^j denotes the quantity ordered by location j at time s. The state vector Q_t^0 consists of $\{q_s^0\}_{t-L_0+1 \leq s \leq t-1}$. Finally, $B_t = \{b_{jt}^s\}_{1 \leq j \leq J, 1 \leq s \leq M}$ is the matrix of unsatisfied backorders at the beginning of period t where b_{jt}^s denotes the number of backorders of age s remaining at retailer j. The parameter M signifies the maximum age. The state variables are related by $I_t^j = z_t^j + \sum_{s=t-L_j+1}^{t-1} q_s^j$.*

Note that the dimension of the state space for V_t^{LB} is a lot smaller than that of V_t^{Exact}. We do not distinguish the backorders of different ages in the state space for V_t^{LB}.

The proof of this lemma is included in the Appendix of this chapter and it makes use of a sample-path argument. The lower bound results from the use of the convex minorant cost function (4.9) in the lower-bound dynamic program not knowing which retailer a unit will eventually get sent to if its aggregate matching time is before its arrival at the central warehouse.

We conclude this subsection by making a transformation identical to that presented in section 3.4 of Chapter 3. The transformation leads to a dynamic program formulation in which the immediate cost functions are convex. Recognize first of all that $\theta_t^j(q_j, I_t^j)$ is equal to $\theta_t^j(q_j + I_t^j, 0)$ by definition and that

$$\sum_{q_j=1}^{r_t^j - I_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, I_t^j))\} = \sum_{q_j=I_t^j+1}^{r_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j - I_t^j, I_t^j))\} = \sum_{q_j=I_t^j+1}^{r_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, 0))\}$$

while

$$\sum_{q_0=1}^{r_t^0 - I_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, I_t^0))\} = \sum_{q_0=I_t^0+1}^{r_t^0} \mathbb{E}\{\Psi_t^0(\theta_t^0(q_0 - I_t^0, I_t^0))\} = \sum_{q_0=I_t^0+1}^{r_t^0} \mathbb{E}\{\Gamma_t^0(\theta_t^0(q_0, 0))\}.$$

Applying property (3.5) for $\sum_{q_j=1}^{r_t^j - I_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, I_t^j))\}$ and $\sum_{q_0=1}^{r_t^0 - I_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, I_t^0))\}$ respectively and using the fact that $\theta_t^j(q_j, 0) = t - 1$ whenever $q_j \leq 0$, we will be able to rewrite the above expressions as

$$\sum_{q_j=1}^{r_t^j - I_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, I_t^j))\} = \sum_{q_j=1}^{r_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, 0))\} - \sum_{q_j=1}^{[I_t^j]^+} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, 0))\}$$
$$+ [-I_t^j]^+ \, \Gamma_t^j(t-1) - [-r_t^j]^+ \, \Gamma_t^j(t-1) \quad (4.11)$$

and

$$\sum_{q_0=1}^{r_t^0 - I_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, I_t^0))\} = \sum_{q_0=1}^{r_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, 0))\} - \sum_{q_0=1}^{[I_t^0]^+} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, 0))\}$$
$$+ [-I_t^0]^+ \, \Psi_t^C(t-1) - [-r_t^0]^+ \, \Psi_t^C(t-1), \quad (4.12)$$

respectively. With the constraint $r_t^0 \geq [I_t^0]^+$ imposed in the lower-bound dynamic program (the central warehouse must always order its echelon inventory position up to at least zero to cover all backorders already existing), we remove $-[-r_t^0]^+ \, \Psi_t^C(t-1)$ from the second expression above because $r_t^0 \geq 0$. These expressions help remove terms that consist of a summation from $q_j = 1$

to $q_j = (r_t^j - D_t^j)$ in the objective function defining V_t^{LB}. Using the following

Transformation for the Lower-Bound Dynamic Program:

$$C_t(I_t, z_t^0, Q_t^0) = V_t^{\text{LB}}(I_t, z_0^t, Q_0^t) + \sum_{q_0=1}^{[I_t^0]^+} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, 0))\} - [-I_t^0]^+ \Psi_t^C(t-1)$$
$$+ \sum_{1 \le j \le J} \sum_{q_j=1}^{[I_t^j]^+} E\{\Gamma_t^j(\theta_t^j(q_j, 0))\} - \sum_{1 \le j \le J} [-I_t^j]^+ \Gamma_t^j(t-1), \quad (4.13)$$

we may use instead the following transformed lower-bound dynamic program:

$$\begin{aligned} C_t(I_t, z_t^0, Q_t^0) \quad = \quad & \min_{r_t \ge I_t} \Bigg\{ \sum_{1 \le j \le J} \Bigg(\sum_{q_j=1}^{r_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, 0))\} + (\pi_{L_j+1}^j - \Gamma_t^j(t-1))[-r_t^j]^+ \\ & - \mathbb{E}\left\{ \sum_{q_j=1}^{[r_t^j - D_t^j]^+} \mathbb{E}\{\Gamma_{t+1}^j(\theta_{t+1}^j(q_j, 0))\} \right\} + \mathbb{E}\{[D_t^j - r_t^j]^+\}\Gamma_{t+1}^j(t) \Bigg) \\ & + \sum_{q_0=1}^{r_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, 0))\} - \left\{ \sum_{q_0=1}^{[r_t^0 - D_t^0]^+} \mathbb{E}\{\Gamma_{t+1}^0(\theta_{t+1}^0(q_0, 0))\} \right\} \\ & + \mathbb{E}\{[D_t^0 - r_t^0]^+\}\Psi_{t+1}^C(t) + h_0 \mathbb{E}\left[r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 \right] \\ & - \sum_{1 \le j \le J} h_j r_t^j + \mathbb{E}\{C_{t+1}(r_t - \vec{D}_t, z_{t+1}^0, Q_{t+1}^0)\} \\ & \text{such that } r_t^0 \ge [I_t^0]^+ \text{and} \sum_{1 \le j \le J} r_t^j \le z_t^0 \Bigg\}. \end{aligned} \quad (4.14)$$

The terminal condition after this transformation turns out to be $C_{T+L_0+1} = 0$. We show this in Lemma 4.4 included in the appendix for this chapter. From here, we work with the transformed lower-bound dynamic program (4.14). The transformation turns out to be crucial, as discussed in the context of the single-location problem with aging backorders in Chapter 3. It permits us to obtain immediate cost functions that are convex after applying a decomposition technique which we will describe in the next section.

4.4 The Clark-and-Scarf Decomposition

In this section, we use the Clark-and-Scarf decomposition approach to break the lower-bound dynamic program (4.14) into multiple single-location problems which are easy to solve. In general, this decomposition gives rise to a lower bound. We start by compressing the notation in (4.14). Define $L_t^j(r_t^j)$ as

$$\begin{aligned} L_t^j(r_t^j) &= \sum_{q_j=1}^{r_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j,0))\} + (\pi_j^{L_j+1} - \Gamma_t^j(t-1))[-r_t^j]^+ \qquad (4.15) \\ &\quad -\mathbb{E}\left\{ \sum_{q_j=1}^{[r_t^j - D_t^j]^+} \mathbb{E}\{\Gamma_{t+1}^j(\theta_{t+1}^j(q_j,0))\} \right\} \\ &\quad +\mathbb{E}\{[D_t^j - r_t^j]^+\}\Gamma_{t+1}^j(t) - h_j r_t^j \qquad (4.16) \end{aligned}$$

for $1 \le j \le J$ and as

$$\begin{aligned} L_t^0(r_t^0) &= \sum_{q_0=1}^{r_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0,0))\} - \mathbb{E}\left\{ \sum_{q_0=1}^{[r_t^0 - D_t^0]^+} \mathbb{E}\{\Psi_{t+1}^C(\theta_{t+1}^0(q_0,0))\} \right\} \\ &\quad + \mathbb{E}\{[D_t^0 - r_t^0]^+\}\Psi_{t+1}^C(t) + h_0 \mathbb{E}\left[r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 \right] \qquad (4.17) \end{aligned}$$

for $j = 0$. Having done this, we may now more succinctly write (4.14) as

$$\begin{aligned} C_t(I_t, z_t^0, Q_t^0) &= \min_{r_t \ge I_t} \left\{ \sum_{1 \le j \le J} L_t^j(r_t^j) + L_t^0(r_t^0) + \mathbb{E}\{C_{t+1}(r_t - \vec{D}_t, z_{t+1}^0, Q_{t+1}^0) \right. \\ &\qquad \left. \text{such that } r_t^0 \ge [I_t^0]^+ \text{and} \sum_{1 \le j \le J} r_t^j \le z_t^0 \right\}. \qquad (4.18) \end{aligned}$$

We show in Lemma 4.5 in the Appendix of this chapter that $L_j(\cdot)$ is convex for $0 \le j \le J$.

We now define the Clark-and-Scarf decomposition of (4.18) as

$$\tilde{C}_t(I_t, z_t^0, Q_t^0) = \sum_{s=t}^{t+L_0-1} \mathbb{E}\Delta_s(z_t^0 + \sum_{u=t+1-L_0}^{s-L_0} q_u^0 - \sum_{u=t}^{s-1} D_u^0) + \sum_{j=0}^{J} C_t^j(I_t^j) \qquad (4.19)$$

where

$$C_t^j(I_t^j) = \min_{r_t^j \geq I_t^j} \left\{ L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j) \right\} \tag{4.20}$$

with

$$C_{T+L_0+1}^j(I_t^j) = 0 \tag{4.21}$$

for the retailers 1 through J, and

$$\Delta_t(z_t^0) = \min_{\sum_{1\leq j\leq J} r_t^j \leq z_t^0} \left\{ \sum_{j=1}^{J} \left\{ L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j) \right. \right.$$
$$\left. \left. - L_t^j(r_t^{j*}) + \mathbb{E}C_{t+1}^j(r_t^{j*} - D_t^j) \right\} \right\} \tag{4.22}$$

where r_t^{j*} is the unconstrained minimizer of problem (4.20). The central-warehouse problem can now be defined as

$$C_t^0(I_t^0) = \min_{r_t^0 \geq [I_t^0]^+} \left\{ L_t^0(r_t^0) + \mathbb{E}\Delta_{t+L_0}(r_t^0 - \sum_{u=t}^{t+L_0-1} D_u^0) + \mathbb{E}C_{t+1}^0(r_t^0 - D_t^0) \right\} \tag{4.23}$$

with the terminal condition

$$C_{T+1}^0(I_{T+1}^0) = 0. \tag{4.24}$$

The solutions to (4.20) through (4.24) have the form of base-stock policies. We state this result in the following theorem.

Theorem 4.2. *The control problems C_t^j for $0 \leq j \leq J$ described by (4.20) to (4.24) have base-stock solutions. In other words, there exists an echelon inventory position level r_t^{j*} for each t and $0 \leq j \leq J$ such that if the echelon inventory position at location j at the beginning of time t, I_t^j, is below this level, it is optimal to order up to r_t^{j*}. Otherwise, it is optimal to do nothing.*

Proof: Because the functions L_t^j, $1 \leq j \leq J$ are convex, by Lemma 4.5, it follows by induction that the functions C_t^j, $1 \leq j \leq J$ are convex. Consequently, the minimization of $L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j)$ is easily done over the domain $r_t^j \geq I_t^j$ for $1 \leq j \leq J$. The unconstrained minimizer r_t^{j*} of $L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j)$ for $1 \leq j \leq J$ may also be obtained to define the function Δ_t, whose convexity easily follows from that of L_t^j and C_{t+1}^j for $1 \leq j \leq J$, as well as the fact that r_t^{j*} is the unconstrained minimizer of $L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j)$. Together with the convexity of L_t^0, the convexity of Δ_t makes the immediate cost function defining C_t^0 convex. By induction, one can then show that C_t^0 is convex also for all t. Therefore, the solutions to (4.20) through (4.24) all have the form of base-stock policies. □

In operating our distribution system, we may solve the following problem at time t to obtain the order-up-to level at the central warehouse and also the allocation quantities for the retailers:

$$\min_{r_t \geq I_t} \left\{ \sum_{1 \leq j \leq J} L_t^j(r_t^j) + L_t^0(r_t^0) + \mathbb{E}\{\tilde{C}_{t+1}(r_t - \vec{D}_t, z_{t+1}^0, Q_{t+1}^0)\} \right.$$
$$\left. \text{such that } r_t^0 \geq [I_t^0]^+ \text{and} \sum_{1 \leq j \leq J} r_t^j \leq z_t^0 \right\}. \quad (4.25)$$

This problem is equivalent to

$$\min_{r_t \geq I_t} \left\{ \sum_{1 \leq j \leq J} L_t^j(r_t^j) + L_t^0(r_t^0) + \sum_{j=0}^{J} \mathbb{E}\{C_{t+1}^j(r_t^j - D_t^j)\} \right.$$
$$+ \mathbb{E}\Delta_{t+L_0}(r_t^0 - \sum_{u=t}^{t+L_0-1} D_u^0) + \sum_{s=t+1}^{t+L_0-1} \mathbb{E}\Delta_s(z_t^0 + \sum_{u=t+1-L_0}^{s-L_0} q_u^0 - \sum_{u=t}^{s-1} D_u^0)$$
$$\left. \text{such that } r_t^0 \geq [I_t^0]^+ \text{and} \sum_{1 \leq j \leq J} r_t^j \leq z_t^0 \right\} \quad (4.26)$$

which is separable into

$$\min_{r_t^0 \geq [I_t^0]^+} \left\{ L_t^0(r_t^0) + \mathbb{E}\Delta_{t+L_0}(r_t^0 - \sum_{u=t}^{t+L_0-1} D_u^0) + \mathbb{E}C_{t+1}^0(r_t^0 - D_t^0) \right\}$$
$$+ \min_{\substack{r_t^j \geq I_t^j, 1 \leq j \leq J \\ \sum_{1 \leq j \leq J} r_t^j \leq z_t^0}} \left\{ \sum_{1 \leq j \leq J} \left[L_t^j(r_t^j) + \sum_{j=1}^{J} \mathbb{E}\{C_{t+1}^j(r_t^j - D_t^j)\} \right] \right\}$$
$$+ \sum_{s=t+1}^{t+L_0-1} \mathbb{E}\Delta_s \left(z_t^0 + \sum_{u=t+1-L_0}^{s-L_0} q_u^0 - \sum_{u=t}^{s-1} D_u^0 \right). \quad (4.27)$$

Note that the retailer problems defined in (4.20) needs to be solved first in order to obtain the functions Δ_t which are needed to obtain the replenishment quantities for the central warehouse.

We now show that this decomposition generally provides a lower bound for the value function in (4.18):

Proposition 4.3. $\tilde{C}_t(I_t, z_t^0, Q_t^0) \leq C_t(I_t, z_t^0, Q_t^0)$.

Proof: We show this result using induction. For $t = T + L_0 + 1$, $\tilde{C}_t(I_t, z_t^0, Q_t^0) = C_t(I_t, z_t^0, Q_t^0) = 0$ and the result is trivially true. Now assume as the induction hypothesis that the result holds for $t + 1$, we get:

$$\begin{aligned} C_t(I_t, z_t^0, Q_t^0) \quad &\geq \quad \min_{\substack{r_t \geq I_t \\ r_t^0 \geq [I_t^0]^+}} \left\{ \sum_{1 \leq j \leq J} L_t^j(r_t^j) + L_t^0(r_t^0) + \mathbb{E}\{\tilde{C}_{t+1}(r_t - \vec{D}_t, z_{t+1}^0, Q_{t+1}^0)\} \right. \\ &\qquad \left. \text{such that} \sum_{1 \leq j \leq J} r_t^j \leq z_t^0 \right\} \\ &= \quad \min_{r_t \geq I_t} \left\{ \sum_{1 \leq j \leq J} L_t^j(r_t^j) + L_t^0(r_t^0) + \sum_{j=0}^{J} \mathbb{E}\{C_{t+1}^j(r_t^j - D_t^j)\} \right. \\ &\qquad + \mathbb{E}\Delta_{t+L_0}(r_t^0 - \sum_{u=t}^{t+L_0-1} D_u^0) + \sum_{s=t+1}^{t+L_0-1} \mathbb{E}\Delta_s(z_t^0 + \sum_{u=t+1-L_0}^{s-L_0} q_u^0 - \sum_{u=t}^{s-1} D_u^0) \\ &\qquad \left. \text{such that } r_t^0 \geq [I_t^0]^+ \text{and} \sum_{1 \leq j \leq J} r_t^j \leq z_t^0 \right\}. \end{aligned}$$

Taking out the constant, we can separate the optimization problem into one involving r_t^0 and one involving r_t^j, $1 \leq j \leq J$. In particular, we get

$$\begin{aligned} C_t(I_t, z_t^0, Q_t^0) &\geq \min_{r_t^0 \geq [I_t^0]^+} \left\{ L_t^0(r_t^0) + \mathbb{E}\Delta_{t+L_0}(r_t^0 - \sum_{u=t}^{t+L_0-1} D_u^0) + \mathbb{E}C_{t+1}^0(r_t^0 - D_t^0) \right\} \\ &+ \min_{\substack{r_t^j \geq I_t^j, 1 \leq j \leq J \\ \sum_{1 \leq j \leq J} r_t^j \leq z_t^0}} \left\{ \sum_{1 \leq j \leq J} \left[L_t^j(r_t^j) + E\{C_{t+1}^j(r_t^j - D_t^j)\} \right] \right\} \\ &+ \sum_{s=t+1}^{t+L_0-1} \mathbb{E}\Delta_s \left(z_t^0 + \sum_{u=t+1-L_0}^{s-L_0} q_u^0 - \sum_{u=t}^{s-1} D_u^0 \right) \\ &= \sum_{s=t+1}^{t+L_0-1} \mathbb{E}\Delta_s \left(z_t^0 + \sum_{u=t+1-L_0}^{s-L_0} q_u^0 - \sum_{u=t}^{s-1} D_u^0 \right) + C_t^0(I_t^0) \\ &+ \min_{\substack{r_t^j \geq I_t^j, 1 \leq j \leq J \\ \sum_{1 \leq j \leq J} r_t^j \leq z_t^0}} \left\{ \sum_{1 \leq j \leq J} \left[L_t^j(r_t^j) + E\{C_{t+1}^j(r_t^j - D_t^j)\} \right] \right\} \end{aligned}$$

We will show that the last term in the last line above is lower-bounded by $\Delta_t(z_t^0) + \sum_{1 \leq j \leq J} C_t^j(I_t^j)$ in Lemma 4.6 included in the Appendix for this chapter. Recognizing this, we can conclude that $C_t(I_t, z_t^0, Q_t^0) \geq \tilde{C}_t(I_t, z_t^0, Q_t^0)$. $\square$

In the next section, we compare the performance of the operating policy driven by this decomposition with the lower bound obtained using the result of Proposition (4.3). Comparing the gap between the average performance of this policy and the lower bound obtained using Proposition (4.3) gives us a sense of how well we are doing relative to the unknown optimal policy.

4.5 Numerical Results

We present the numerical results obtained using our proposed policy (CS) which applies Clark-and-Scarf's decomposition as described by (4.27). We compare (CS) with three other benchmark policies which are described below.

Myopic policy based on the aggregate-matching cost-accounting mechanism (MYO) In this benchmark policy, we make use of (4.27) but we replace C^j_{t+1} with 0 where they appear. The penalty function is defined using (4.22) but with r^{j*}_t replaced by the unconstrained minimizer of the immediate cost component, $L^j_t(r^j_t)$, of the retailer problem given in (4.20). This policy is myopic in the sense that we consider only the total expected cost incurred by the units we procure in this time period. We neglect the effect of our decisions on the starting echelon inventory positions in the next period which impact the total expected cost incurred by units we procure in the next time period.

Myopic policy based on standard cost accounting (STA) We consider the effect of the decisions made in period t on the standard one-period expected cost incurred at the end of period $t + L_0$ at the central warehouse and at the end of period $t + L_j$ at retailer j for $1 \leq j \leq J$. In particular, for retailer j, we charge the one-period cost $\mathbb{E}\left\{h_j\left(r^j_t - \sum_{s=t}^{t+L_j} D^j_s\right)^+ + \sum_{i=1}^{M}(\pi^j_i \wedge \pi^j_{L_j+1})\tilde{b}^{i,j}_{t+L_j}\right\}$ where $\tilde{b}^{i,j}_{t+L_j}$ is the number of backorders of age i left at retailer j at the end of period $t + L_j$. Note that for backorders of ages older than $L_j + 1$, we charge the per-period cost of $\pi^j_{L_j+1}$ at the end of period $t + L_j$ at the level of the retailers. The residual charges will be considered in the one-period cost function for the central warehouse. As described in section 3.5, we can rewrite the term $\sum_{i=1}^{M}(\pi^j_i \wedge \pi^j_{L_j+1})\tilde{b}^{i,j}_{t+L_j}$ as

$$\sum_{i=1}^{M}(\pi^j_i \wedge \pi^j_{L_j+1})\tilde{b}^{i,j}_{t+L_j} = \sum_{i=1}^{L_j}(\pi^j_i - \pi^j_{i-1})\sum_{k=i}^{M}\tilde{b}^{k,j}_{t+L_j} + (\pi^j_{L_j+1} - \pi^j_{L_j})\sum_{k=L_j+1}^{M}\tilde{b}^{k,j}_{t+L_j} \tag{4.28}$$

where $\pi^j_0 = 0$ and

$$\sum_{k=i}^{M}\tilde{b}^{k,j}_{t+L_j} = \left(\sum_{s=t}^{t+L_j+1-i} D^j_s - r^j_t\right)^+ \tag{4.29}$$

for $1 \leq i \leq L_j + 1$. For the central warehouse, in addition to a lower bound for the expected residual charges incurred for backorders older than $L_j + 1$ periods at the end of $t + L_0 + L_j$ at retailer j for $1 \leq j \leq J$ which result from demands occurring between t and $t + L_0 - 1$, we also include the expected holding cost incurred at the central warehouse at the end of $t + L_0$. A lower-bound for the residual backorder charges incurred at $t + L_0 + L_j$ for $1 \leq j \leq J$ can be obtained by observing that

$$\sum_{1 \leq j \leq J} \sum_{i=L_j+2}^{L_0+L_j+1} (\pi_i^j - \pi_{L_j+1}^j) \tilde{b}_{t+L_0+L_j}^{i,j} \tag{4.30}$$

$$= \sum_{1 \leq j \leq J} \sum_{i=L_j+2}^{L_0+L_j+1} (\pi_i^j - \pi_{i-1}^j) \sum_{k=i}^{L_0+L_j+1} \tilde{b}_{t+L_0+L_j}^{k,j} \tag{4.31}$$

$$\geq \sum_{i=1}^{L_0} \min_{1 \leq j \leq J} (\pi_{L_j+1+i}^j - \pi_{L_j+1}^j) \sum_{1 \leq j \leq J} \sum_{k=i+L_j+1}^{L_0+L_j+1} \tilde{b}_{t+L_0+L_j}^{k,j} \tag{4.32}$$

$$= \sum_{i=1}^{L_0} \min_{1 \leq j \leq J} (\pi_{L_j+1+i}^j - \pi_{L_j+1}^j) \left(\sum_{s=t}^{t+L_0-i} D_s^0 - r_t^0 \right)^+ \tag{4.33}$$

Putting it together, we use the following one-period cost in period t under standard cost accounting:

$$\sum_{i=1}^{L_0} \min_{1 \leq j \leq J} (\pi_{L_j+1+i}^j - \pi_{L_j+i}^j) \mathbb{E} \left(\sum_{s=t}^{t+L_0-i} D_s^0 - r_t^0 \right)^+ + \mathbb{E} h_0 \left(r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 - h_j r_j^t \right)$$
$$+ \sum_{1 \leq j \leq J} \mathbb{E} \left[h_j \left(r_t^j - \sum_{s=t}^{t+L_j} D_s^j \right)^+ + \sum_{i=1}^{L_j+1} (\pi_i^j - \pi_{i-1}^j) \left(\sum_{s=t}^{t+L_j+1-i} D_s^j - r_t^j \right)^+ \right]. \tag{4.34}$$

The myopic policy based on standard cost accounting makes use of this one-period function. Note that this is the one-period cost function one expects to get extending the results of Huh et al. (2011) to a two-echelon inventory distribution system with aging backorders.

Heuristic policy with $\Psi_t^C(\cdot)$ replaced by a convex combination of $\Psi_t^j(\cdot)$, $1 \leq j \leq J$ (HEU) In this benchmark policy, instead of using the convex minorant function $\Psi_t^C(\cdot)$ in the definition of $L_t^0(\cdot)$ which appears in (4.17), we use a weighted average of the functions $\Psi_t^j(\cdot)$. Recall that the function $\Psi_t^j(\theta_0^t(q_0, 0))$ captures a lower bound for the non-linear residual backorder charges incurred for some backorder at retailer j should the echelon inventory position brought up to by the q_0-th unit ordered by the central warehouse gets cleared by the aggregate demand at the retailers starting at time t before the q_0-th unit arrives at the central warehouse. In order to get an overall lower bound, we can only use the convex minorant of the functions $\Psi_t^j(\cdot)$ since we do not know which retailer this backorder will be at at the time of purchase. In (HEU), we use an average of the functions $\Psi_t^j(\cdot)$ weighted by estimates of the likelihood of this backorder being at retailer j. In particular, we use as the weights $\frac{\sum_{s=(t-1)\vee 1}^{(t+L_0-2)\wedge T} \mathbb{E}[D_s^j]}{\sum_{1 \le j \le J} \sum_{s=(t-1)\vee 1}^{(t+L_0-2)\wedge T} \mathbb{E}[D_s^j]}$ so that the retailers with higher demand averages between $t-1$ and $t+L_0-2$ (the set of values of θ_t^0 that make Ψ_t^j non-zero) have higher weights. Note that by doing this, we can no longer show that our decomposition is an overall lower bound.

In all of our test problems, the time horizon consists of 50 periods with non-zero Poisson demands at the retailers. In all test cases, there are two retailers and a central warehouse. The per-period per-unit holding cost is 0.2 at the central warehouse and 1 at the two retailers. We consider three different combinations of procurement lead times for the central warehouse and the retailers. In the first case, the procurement lead times of the central warehouse and the 2 retailers are all equal to 5. In the second case, we change the procurement lead time of the central warehouse to 10 and in the last case, we fix the procurement lead time of the central warehouse at 10 while changing the procurement lead times of the retailers to 1. We consider three types of base-case demand profiles. In base-case demand profile “a”, the expected demands for these 50 periods are equal to $5 + \sin(\pi t/25)$ for $1 \le t \le 50$. In base-case demand profile “b”, we generate the means for these 50 periods using a uniform distribution on $\{3, 4, 5, 6, 7, 8\}$. We generate them only once, at the beginning of a test problem, and they remain fixed thereafter. In base-case demand profile "c", the demand is stationary with mean 5 at the retailer.

For backorder costs, we consider three different base cases of increasing per-period backorder costs. In base case 1, $\pi_j^i = 2 + 2(i-1)$ so the per-period backorder cost increases in a linear fashion for both retailers. In base case 2, we use $\pi_j^i = (1.5)^i$ so that the per-period backorder cost increases exponentially for both retailers. In the last case, we consider a constant per-period backorder cost of $\pi_j^i = 10$. This last case is included so that we see the gap between the lower bound and the average performance of Clark-and-Scarf's decomposition under the ordinary set-up with no aging backorders in an inventory distribution system.

We consider 5 different types of problems. In the first type of problem, which we call $(D, D), (\pi, \pi)$, the two retailers are completely identical and they both have the base-case demand profiles and the base-case backorder cost structures. The second type of problem, which we call $(D, D), (\pi, 2\pi)$, the two retailers have identical demand profiles but one retailer has twice the per-period backorder costs described by the base-case. In problem $(D, 2D), (\pi, \pi)$, the expected per-period demands of one of the retailers are equal to two times those described in the base-case. In problem $(D, 2D), (2\pi, \pi)$, the retailer with dominating per-period backorder costs has lower expected per-period demands. Finally, we consider cases where the retailer with dominating per-period backorder costs has higher expected per-period demands in problem $(D, 2D), (\pi, 2\pi)$. Motivating these 5 different problem types is the fact that Ψ_t^C is exactly half of Ψ_t^j for one of the retailers in the case that one retailer has twice the per-period backorder costs. Depending on the demands expected at this retailer, the lower-bound for the residual backorder costs charged at the central warehouse could significantly underestimate those incurred in actuality at the retailers for backorders older than $L_j + 1$ periods. We want to see how well (CS) performs relative to the lower-bound in these cases.

The results of our numerical study are included in the first appendix of this chapter. The results for each of the five problem types are included in Tables (4.1) through (4.5). The first column gives the parameters used for each test case. The first entry of the 4-tuple corresponds to the base demand profile used. The second entry of the 4-tuple gives the procurement lead time of the central warehouse and the third entry gives the identical procurement lead times of the two retailers. The last entry gives the base increasing per-period backorder costs. The column LB gives the lower bound computed using the result of Proposition (4.3). The last four columns illustrate the percentage differences between (CS) and LB, as well as those between (STA), (MYO), (HEU) and (CS) respectively. A checkmark (✓) is included for those cases where the percentage difference is not statistically different from 0, at the 95% confidence level.

Note that (HEU) in general provides no advantages over (CS) for all five problem types including the one in which more demands are expected to come from the retailer with higher per-period backorder costs, where one might expect (HEU) to perform better. We point out that in all the test cases considered, our proposed (CS) policy is never more than 3.5% greater than the LB. Even extending the procurement lead time of the central warehouse from 5 to 10 while keeping those of the retailers at 5 does not seem to have a dramatic effect on the performance of the (CS) policy. This also seems to be the case when we keep the lead time of the central warehouse at 10 while changing those of the retailers to 1 such that a lot of the residual costs incurred at the retailers are charged at the central warehouse via the convex minorant function. This is what we observe for all the problem types, even in the case where the retailer with higher per-period backorder costs also has higher expected demands.

It is also worth pointing out that the incorporation of increasing per-period costs does not seem to significantly deteriorate the performance of Clark-and-Scarf's decomposition in our numerical study since the gaps observed in test cases where the per-period costs do not increase (type 3 under Π in the tables) are not very different in size compared with those test cases where the per-period costs do increase (type 1 and type 2 under Π).

Our numerical study shows the (CS) policy performs well for the test cases we considered. We cannot, however, conclude that the (CS) policy will always exhibit this kind of performance. This is in view of situations where Clark-and-Scarf decomposition performs poorly (large percentage gap between the average cost incurred by (CS) and the lower bound) even under the standard assumption of non-increasing per-period backorder costs. One can see, for example, the numerical study in Kunnumkal and Topaloglu (2008). Nonetheless, Clark-and-Scarf's decomposition still provides us with a viable alternative which can be used to deal with inventory distribution systems with aging backorders in a computationally tractable manner.

4.6 Appendix for Chapter 4 with Tables of Numerical Results

Problem Type $(D, D), (\pi, \pi)$

Problem	Tot. Exp. Cost						STA	MYO	HEU
$(D, L_{\mathrm{CW}}, L_{\mathrm{ret.}}, \Pi)$	STA	MYO	CS	HEU	LB	CS-LB(%)	- CS(%)		
(a,5,5,1)	908	908	885	885	859	2.97	3.15	3.12	0.00 ✓
(a,5,5,2)	794	794	771	771	749	2.88	3.37	3.38	0.00 ✓
(a,5,5,3)	1335	1336	1308	1308	1274	2.72	2.55	2.63	0.00 ✓
(b,5,5,1)	874	874	850	850	828	2.73	3.24	3.30	0.00 ✓
(b,5,5,2)	749	749	729	729	721	1.14 ✓	3.14	3.07	0.00 ✓
(b,5,5,3)	1280	1280	1246	1246	1235	0.92 ✓	3.23	3.23	0.00 ✓
(c,5,5,1)	872	871	849	849	830	2.29	3.22	3.03	0.00 ✓
(c,5,5,2)	751	751	731	731	726	0.63 ✓	3.12	3.09	0.00 ✓
(c,5,5,3)	1285	1285	1251	1251	1243	0.65 ✓	3.24	3.22	0.00 ✓
(a,10,5,1)	1068	1068	1014	1014	984	2.99	6.47	6.47	0.00 ✓
(a,10,5,2)	938	937	890	890	868	2.59	6.21	6.01	0.00 ✓
(a,10,5,3)	1540	1541	1472	1472	1427	3.16	5.73	5.75	0.00 ✓
(a,10,1,1)	609	612	575	575	574	0.18 ✓	6.26	6.93	0.00 ✓
(a,10,1,2)	541	544	511	511	510	0.20 ✓	6.49	7.06	0.00 ✓
(a,10,1,3)	947	948	895	895	892	0.42 ✓	6.43	6.44	0.00 ✓

Table 4.1: Total expected costs incurred by STA, MYO, CS and HEU for problem type $(D, D), (\pi, \pi)$ where the two retailers are identical.

Problem Type $(D, D), (\pi, 2\pi)$

Problem $(D, L_{\text{CW}}, L_{\text{ret.}}, \Pi)$	Tot. Exp. Cost STA	MYO	CS	HEU	LB	CS-LB(%)	STA - CS(%)	MYO	HEU
(a,5,5,1)	990	990	969	969	951	1.97	2.73	2.73	0.00 ✓
(a,5,5,2,)	878	878	857	857	841	1.95	2.88	2.92	-0.02 ✓
(a,5,5,3)	1430	1431	1398	1398	1369	2.10	2.97	3.09	0.00 ✓
(b,5,5,1)	961	961	937	937	930	0.74 ✓	3.08	3.03	0.00 ✓
(b,5,5,2)	847	846	829	829	823	0.70 ✓	2.73	2.61	0.00 ✓
(b,5,5,3)	1383	1386	1351	1351	1338	0.99 ✓	2.98	3.19	0.00 ✓
(c,5,5,1)	973	973	950	949	935	1.54	3.01	3.04	0.00 ✓
(c,5,5,2)	856	856	833	832	827	0.73 ✓	3.21	3.29	-0.02 ✓
(c,5,5,3)	1400	1401	1362	1362	1344	1.31 ✓	3.44	3.56	0.00 ✓
(a,10,5,1)	1127	1133	1072	1071	1056	1.47 ✓	6.19	6.79	-0.01 ✓
(a,10,5,2)	1001	1001	949	950	942	0.80 ✓	6.10	6.13	0.09
(a,10,5,3)	1574	1574	1501	1501	1476	1.71	5.99	5.97	0.00 ✓
(a,10,1,1)	648	647	614	614	617	-0.43 ✓	5.91	5.89	0.08 ✓
(a,10,1,2)	573	573	550	550	552	-0.37 ✓	4.61	4.59	0.20
(a,10,1,3)	955	955	917	917	921	-0.50 ✓	4.65	4.65	0.00 ✓

Table 4.2: Total expected costs incurred by STA, MYO, CS and HEU for problem type $(D, D), (\pi, 2\pi)$ where one retailer has twice the per-period backorder cost as the other retailer.

Problem Type $(D, 2D), (\pi, \pi)$

Problem $(D, L_{\text{CW}}, L_{\text{ret.}}, \Pi)$	Tot. Exp. Cost STA	MYO	CS	HEU	LB	CS-LB(%)	STA - CS(%)	MYO	HEU
(a,5,5,1)	1055	1055	1022	1022	1014	0.73 ✓	3.68	3.68	0.00 ✓
(a,5,5,2,)	926	923	895	895	889	0.69 ✓	3.82	3.43	0.00 ✓
(a,5,5,3)	1600	1600	1549	1549	1537	0.79 ✓	3.73	3.77	0.00 ✓
(b,5,5,1)	1022	1023	999	999	981	1.81	2.86	2.95	0.00 ✓
(b,5,5,2)	899	899	876	876	860	1.84	2.94	2.92	0.00 ✓
(b,5,5,3)	1553	1551	1513	1513	1484	1.95	3.20	3.00	0.00 ✓
(c,5,5,1)	1037	1036	1008	1008	985	2.25	3.38	3.30	0.00 ✓
(c,5,5,2)	910	910	887	887	862	2.95	2.97	3.01	0.00 ✓
(c,5,5,3)	1568	1573	1533	1533	1490	2.88	2.88	3.27	0.00 ✓
(a,10,5,1)	1194	1195	1142	1142	1133	0.81 ✓	5.48	5.65	0.00 ✓
(a,10,5,2)	1063	1064	1012	1012	1003	0.91 ✓	5.77	6.00	0.00 ✓
(a,10,5,3)	1738	1741	1658	1658	1658	-0.02 ✓	5.75	5.89	0.00 ✓
(a,10,1,1)	723	728	686	686	678	1.20 ✓	6.07	6.80	0.00 ✓
(a,10,1,2)	653	654	619	619	614	0.79 ✓	6.13	6.25	0.00 ✓
(a,10,1,3)	1125	1125	1060	1060	1050	0.94 ✓	6.71	6.72	0.00 ✓

Table 4.3: Total expected costs incurred by STA, MYO, CS and HEU for problem type $(D, 2D), (\pi, \pi)$ where one retailer has twice the per-period expected demand as the other retailer.

Problem Type $(D, 2D), (2\pi, \pi)$

Problem $(D, L_{\text{CW}}, L_{\text{ret.}}, \Pi)$	Tot. Exp. Cost STA	MYO	CS	HEU	LB	CS-LB(%)	STA - CS(%)	MYO	HEU
(a,5,5,1)	1096	1098	1072	1072	1065	0.65 ✓	2.72	2.88	0.00 ✓
(a,5,5,2,)	969	969	948	948	944	0.42 ✓	2.62	2.59	0.01 ✓
(a,5,5,3)	1602	1601	1562	1562	1554	0.54 ✓	3.13	3.09	0.00 ✓
(b,5,5,1)	1118	1116	1095	1095	1083	1.12 ✓	2.63	2.46	0.00 ✓
(b,5,5,2)	1003	1002	980	980	960	2.10	2.59	2.50	0.00 ✓
(b,5,5,3)	1657	1658	1619	1619	1587	2.05	2.91	2.93	0.00 ✓
(c,5,5,1)	1138	1138	1112	1112	1085	2.54	2.87	2.88	-0.02
(c,5,5,2)	1007	1008	984	984	966	1.88	2.82	2.87	-0.02 ✓
(c,5,5,3)	1657	1657	1617	1617	1595	1.36 ✓	3.08	3.12	0.00 ✓
(a,10,5,1)	1341	1341	1281	1281	1267	1.15 ✓	5.65	5.62	-0.01 ✓
(a,10,5,2)	1179	1180	1130	1129	1119	0.91 ✓	5.31	5.40	0.00 ✓
(a,10,5,3)	1901	1900	1822	1822	1800	1.23 ✓	5.41	5.35	0.00 ✓
(a,10,1,1)	805	809	764	764	758	0.87 ✓	5.74	6.35	0.07
(a,10,1,2)	727	731	687	687	680	1.14 ✓	6.05	6.66	0.09
(a,10,1,3)	1217	1217	1157	1157	1145	1.07 ✓	5.81	5.83	0.00 ✓

Table 4.4: Total expected costs incurred by STA, MYO, CS and HEU for problem type $(D, 2D), (2\pi, \pi)$ where the retailer with dominating per-period backorder costs has lower expected per-period demands.

Problem Type $(D, 2D), (\pi, 2\pi)$

Problem $(D, L_{\text{CW}}, L_{\text{ret.}}, \Pi)$	Tot. Exp. Cost STA	MYO	CS	HEU	LB	CS-LB(%)	STA - CS(%)	MYO	HEU
(a,5,5,1)	1172	1174	1145	1145	1124	1.90	3.10	3.24	-0.02 ✓
(a,5,5,2,)	1046	1046	1018	1017	1001	1.69	3.19	3.21	-0.01 ✓
(a,5,5,3)	1682	1682	1638	1638	1608	1.87	3.51	3.50	0.00 ✓
(b,5,5,1)	1168	1167	1143	1143	1126	1.51	2.96	2.89	-0.02 ✓
(b,5,5,2)	1051	1052	1029	1028	1007	2.13	2.67	2.77	-0.03 ✓
(b,5,5,3)	1695	1696	1665	1665	1630	2.12	2.79	2.82	0.00 ✓
(c,5,5,1)	1180	1182	1154	1154	1128	2.33	2.88	3.00	-0.01 ✓
(c,5,5,2)	1045	1044	1015	1015	1006	0.91 ✓	3.32	3.25	-0.01 ✓
(c,5,5,3)	1689	1689	1646	1646	1630	0.96 ✓	3.29	3.32	0.00 ✓
(a,10,5,1)	1390	1390	1332	1331	1316	1.16 ✓	5.50	5.49	-0.01 ✓
(a,10,5,2)	1247	1247	1191	1190	1181	0.85 ✓	5.75	5.75	0.06 ✓
(a,10,5,3)	1940	1940	1864	1864	1845	1.07 ✓	5.27	5.23	0.00 ✓
(a,10,1,1)	841	841	798	797	787	1.32 ✓	6.09	6.10	0.07 ✓
(a,10,1,2)	760	760	719	718	707	1.64 ✓	6.36	6.38	0.12 ✓
(a,10,1,3)	1239	1239	1179	1179	1167	0.95 ✓	5.88	5.87	0.00 ✓

Table 4.5: Total expected costs incurred by STA, MYO, CS and HEU for problem type $(D, 2D), (\pi, 2\pi)$ where the retailer with dominating per-period backorder costs also has two times the expected per-period demands.

4.7 Appendix for Chapter 4 with Omitted Results

We include the omitted proofs from Chapter 4 in this appendix.

Lemma 4.1. *$V_t^{LB}(I_t, z_t^0, Q_t^0)$ as described by (4.10) and $V_t^{Exact}(z_t, B_t, Q_t)$ as described by (4.4) satisfy $V_t^{LB}(I_t, z_t^0, Q_t^0) \leq V_t^{Exact}(z_t, B_t, Q_t)$. Here $z_t = (z_t^0, z_t^1, ..., z_t^J)$ is the vector of echelon net inventory and $I_t = (I_t^0, I_t^1, ..., I_t^J)$ is the vector of echelon inventory positions. Also, $Q_t = \{q_s^j\}_{t-L_j+1 \leq s \leq t-1, 0 \leq j \leq J}$ is the matrix of on-order inventory where q_s^j denotes the quantity ordered by location j at time s. The state vector Q_t^0 consists of $\{q_s^0\}_{t-L_0+1 \leq s \leq t-1}$. Finally, $B_t = \{b_{jt}^s\}_{1 \leq j \leq J, 1 \leq s \leq M}$ is the matrix of unsatisfied backorders at the beginning of period t where b_{jt}^s denotes the number of backorders of age s remaining at retailer j. The parameter M signifies the maximum age. The state variables are related by $I_t^j = z_t^j + \sum_{s=t-L_j+1}^{t-1} q_s^j$.*

Proof of Lemma 4.1:

We start by restating the definitions of $V_t^{\text{Exact}}(z_t, B_t, Q_t)$ and $V_t^{\text{LB}}(I_t, z_t^0, Q_t^0)$. By (4.10),

$$\begin{aligned} V_t^{\text{LB}}(I_t, z_t^0, Q_t^0) \quad = \quad & \min_{\substack{r_t \geq I_t, \\ r_t^0 \geq [I_t^0]^+, \\ \sum_{1 \leq j \leq J} r_t^j \leq z_t^0}} \left\{ \sum_{1 \leq j \leq J} \left(\sum_{q_j=1}^{r_t^j - I_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, I_t^j))\} + \pi_{L_j+1}^j [-r_t^j]^+ \right) \right. \\ & + \sum_{q_0=1}^{r_t^0 - I_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, I_t^0))\} + \mathbb{1}_{\{t \leq T+1\}} h_0 \mathbb{E}\left[r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 \right] \\ & \left. - \sum_{1 \leq j \leq J} h_0 r_t^j + \mathbb{E}\{V_{t+1}^{\text{LB}}(r_t - D_t, z_{t+1}^0, Q_{t+1}^0)\} \right\}, \end{aligned}$$

and by (4.4),

$$V_t^{\text{Exact}}(z_t, B_t, Q_t) = \min_{\substack{r_t \geq I_t, \\ \sum_{1\leq j\leq J} r_t^j \leq z_t^0}} \Bigg\{ -\sum_{1\leq j\leq J} h_0 r_t^j + \mathbb{1}_{\{t\leq T+1\}} h_0 \mathbb{E}\left(r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 \right)$$
$$+ \sum_{1\leq j\leq J} \mathbb{E} h_j (r_t^j - \sum_{s=t}^{t+L_j} D_s^j)^+ + \mathbb{E} \sum_{1\leq j\leq J} \sum_{s=1}^{M} \pi_s^j b_{j,t+L_j+1}^s + V_{t+1}^{\text{Exact}}(z_{t+1}, B_{t+1}, Q_{t+1}) \Bigg\}.$$

The boundary condition for (4.4) is $V_{T+L_0+2}^{\text{Exact}} = 0$ and the boundary condition for (4.10) is $V_{T+L_0+1}^{\text{LB}}(I_t, z_t^0, Q_t^0) = \sum_{1\leq j\leq J} \Pi_{L_j+1}^j [-I_{T+L_0+1}^j]^+$.

Recall the definitions

$$\Gamma_t^j(\cdot) = h_j(\cdot - (t + L_j))^+ + \sum_{i=1}^{t+L_j-\cdot} \pi_i^j \tag{4.35}$$

and

$$\Psi_t^C(\cdot) = \text{conv} \min{}_{1\leq j\leq J} \left\{ \sum_{i=L_j+2}^{t+L_0+L_j-\cdot} \left[\pi_i^j - \pi_{L_j+1}^j \right] \right\}. \tag{4.36}$$

Furthermore, the stopping-time random variable $\theta_t^j(q_j, I_t^j)$ is as defined in (4.5).

We seek to show that $V_t^{\text{LB}} \leq V_t^{\text{Exact}}$. We begin by comparing their immediate cost functions. Ignoring the terms that are identical for both functions, we have remaining in the immediate cost function for V_t^{LB}

$$\sum_{1\leq j\leq J} \left(\sum_{q_j=1}^{r_t^j - I_t^j} \mathbb{E}\{\Gamma_t^j(\theta_t^j(q_j, I_t^j))\} + \pi_{L_j+1}^j [-r_t^j]^+ \right) + \sum_{q_0=1}^{r_t^0 - I_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, I_t^0))\}, \tag{4.37}$$

and remaining in the immediate cost function for V_t^{Exact}

$$\sum_{1\leq j\leq J} \mathbb{E} h_j (r_t^j - \sum_{s=t}^{t+L_j} D_s^j)^+ + \mathbb{E} \sum_{1\leq j\leq J} \sum_{s=1}^{M} \pi_s^j b_{j,t+L_j+1}^s. \tag{4.38}$$

We show the result of the lemma for $t = 1$ without loss of generality. We proceed by using a sample-path argument which assumes that the demands are known in advance. The sample-path argument further assumes that the order quantities and the matchings between demands and inventory are given by the solution to V_t^{Exact} (which is also feasible for V_t^{LB}) and are known in advance. We define the following to use in the rest of our proof:

- Let $\hat{b}_{jt}^s$ be the backorders due to D_t^j which get satisfied at the age of s.
- Let $x_t^j,\ 0 \leq j \leq J$, be the on-hand inventory at location j at time t after receiving the order placed at $t - L_j$, before the new procurement decision is made and before demand for period t is realized.
- Let $\mathcal{Q}_t^j$, be the set of all units successfully ordered by location j at time t. (Every unit in these sets has a unique identifier k. We will be using such qualifier as $k \in \mathcal{Q}_t^j$.)
- Let τ_k be the arrival time of the demand that gets matched with unit k under the matching rule given by the solution to V_t^{Exact}.
- Let η_k be the retailer at which the matching demand for unit k occurs under the matching rule given by the solution to V_t^{Exact}.

Using the above definitions, the sum of (4.38) over $t = 1$ to $t = T + L_0 + 1$ along each sample path is equal to:

$$\sum_{1 \leq j \leq J} \sum_{t=1}^{T} \sum_{s=1}^{M} \Pi_s^j \hat{b}_{jt}^s + \sum_{1 \leq j \leq J} \sum_{t=L_j+1}^{T+L_0+L_j+1} h_j (x_t^j - D_t^j)^+. \tag{4.39}$$

It is clear that charging the total backorder cost for a demand incrementally for each period it remains as a backorder, as in (4.38), is the same as charging this total cost in its entirety when the demand is realized, as in (4.39), as long as the system has no backorders at the beginning of period 1. Otherwise, (4.39) is a lower bound since we are ignoring the costs associated with backorders that exist in the system at the beginning of period 1.

As for the sum of (4.37) over $t = 1$ to $t = T + L_0 + 1$ under the matching rule given by the solution to V_t^{LB}, it is upper bounded by

$$\begin{aligned}
&\sum_{1\le j\le J}\sum_{t=1}^{T+L_0+1}\left\{\sum_{k\in\mathcal{Q}_t^j}\Gamma_t^j(\tau_k)+\pi_{L_j+1}^j[-r_t^j]^+\right\}+\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}_u^0}\Psi_u^C(\tau_k\vee(u-1))\\
=&\sum_{1\le j\le J}\sum_{u=1}^{T+1+L_0}\left\{\sum_{k\in\mathcal{Q}_u^j}\Pi^j_{(L_j+u-[\tau_k\vee(u-1)])^+}+h_j\left(\tau_k-(u+L_j)\right)^+\right.\\
&\left.\qquad+\pi_{L_j+1}^j[-r_u^j]^+\right\}+\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}_u^0}\Psi_u^C(\tau_k\vee(u-1)).
\end{aligned}\tag{4.40}$$

Note that if a unit ordered by the central warehouse at time t has $t-1\le\theta_t^0\le t+L_0-2$ as defined in (4.5) such that $\Psi_t^C(\theta_t^0)$ is non-zero as defined in (4.8) and (4.9), a backorder existing at one of the retailers at time θ_t^0, whose arrival time is less than or equal to θ_t^0, will need to wait for a unit ordered by the central warehouse at $u\ge t$. The matching unit will incorporate a cost of $\Psi_u^C(\tau_k\vee(u-1))\ge\Psi_t^C(\theta_t^0)$ in V_t^{LB} since $u\ge t$ and $\tau_k\le\theta_t^0$. Expression (4.40) is also an upper bound because τ_k is given by the optimal solution to V_t^{Exact}, despite being feasible for the problem defined by V_t^{LB}.

It remains to show that (4.40) is less than or equal to (4.39). We start by showing that

$$\begin{aligned}
&\sum_{1\le j\le J}\sum_{t=1}^{T}\sum_{s=1}^{M}\Pi_s^j\hat{b}_{jt}^s\\
\ge&\sum_{1\le j\le J}\sum_{u=1}^{T+1+L_0}\left(\sum_{k\in\mathcal{Q}_u^j}\Pi^j_{(L_j+u-[\tau_k\vee(u-1)])^+}+\pi_{L_j+1}^j[-r_u^j]^+\right)\\
&+\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}_u^0}\Psi_u^C(\tau_k\vee(u-1))
\end{aligned}$$

Because $\hat{b}_{jt}^s$ is the total of number of backorders due to D_t that get satisfied at time $t+s$ at the age of s, their inventory units must have been ordered by the retailer at time $t+s-L_j$. In other words, $\hat{b}_{jt}^s=\sum_{k\in\mathcal{Q}^j_{t+s-L_j}}\mathbb{1}_{\{\tau_k=t\}}$. We sum over all the units ordered in period $t+s-L_j$ whose matching demand occurs in period t. Substituting this in, we get that:

$$\sum_{1\le j\le J}\sum_{t=1}^{T}\sum_{s=1}^{M}\Pi_s^j\hat{b}_t^s = \sum_{1\le j\le J}\sum_{t=1}^{T}\sum_{s=1}^{M}\sum_{k\in\mathcal{Q}_{t+s-L_j}^j}\Pi_s^j\mathbb{1}_{\{\tau_k=t\}} \tag{4.41}$$

$$= \sum_{1\le j\le J}\sum_{t=1}^{T}\left(\sum_{s=1}^{L_j+1}\sum_{k\in\mathcal{Q}_{t+s-L_j}^j}\Pi_s^j\mathbb{1}_{\{\tau_k=t\}}\right. \tag{4.42}$$

$$\left.+\sum_{s=L_j+2}^{M}\sum_{k\in\mathcal{Q}_{t+s-L_j}^j}\Pi_s^j\mathbb{1}_{\{\tau_k=t\}}\right). \tag{4.43}$$

We analyze (4.42) and (4.43) separately. We start with (4.42) and rewrite it as

$$\sum_{1\le j\le J}\sum_{t=1}^{T}\sum_{u=t+1-L_j}^{t+1}\sum_{k\in\mathcal{Q}_u^j}\Pi_{L_j-t+u}^j\mathbb{1}_{\{\tau_k=t\}}$$

$$=\sum_{1\le j\le J}\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}_u^j}\sum_{t=u-1}^{u-1+L_j}\Pi_{L_j-t+u}^j\mathbb{1}_{\{\tau_k=t\}}$$

$$=\sum_{1\le j\le J}\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}_u^j}\Pi_{(L_j-(\tau_k-u))^+}^j\mathbb{1}_{\{\tau_k\ge u-1\}}. \tag{4.44}$$

The first line follows using the substitution $u=t+s-L_j$. The equality that follows is obtained by switching the order of summation. The last equality follows easily by explicitly evaluating the inner-most summation.

We now turn to (4.43). Note that if $k\in\mathcal{Q}_{t+s-L_j}^j$, the same unit k may also belong to some set $\mathcal{Q}_u^0$ where $u\le t+s-L_j-L_0$. (It could also be that this unit is already in the system at the beginning of period 1.) We lower bound this term by focusing exclusively on those units which are in $\mathcal{Q}_{t+s-L_j-L_0}^0$. Furthermore, we change the upper limit of the summation over s to $s=L_0+L_j+1$ which is less than or equal to M. Both of these relaxations are related to the assumption we make in charging a lower-bound for the residual backorder costs at the central warehouse in setting up the immediate cost function found in the lower-bound dynamic program. Therefore, we can lower bound (4.43) using

$$\begin{aligned}
&\sum_{1\le j\le J}\sum_{t=1}^{T}\sum_{s=L_j+2}^{M}\sum_{k\in\mathcal{Q}^j_{t+s-L_j}}\Pi^j_s\mathbb{1}_{\{\tau_k=t\}}\\
\ge\;&\sum_{1\le j\le J}\sum_{t=1}^{T}\sum_{s=L_j+2}^{L_0+L_j+1}\sum_{k\in\mathcal{Q}^0_{t+s-L_0-L_j}}\Pi^j_s\mathbb{1}_{\{\tau_k=t,\eta_k=j\}} &(4.45)\\
\ge\;&\sum_{1\le j\le J}\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}^0_u}\Pi^j_{L_0+L_j-(\tau_k-u)}\mathbb{1}_{\{u-1\le\tau_k\le u+L_0-2,\eta_k=j\}}. &(4.46)
\end{aligned}$$

Note that after we switch from $\mathcal{Q}^j_{t+s-L_j}$ to $\mathcal{Q}^0_{t+s-L_0-L_j}$, we need to add to the indicator function the condition $\eta_k = j$ which tells us which retailer unit k gets sent to. The second inequality is obtained using the same steps that took us from expression (4.42) to expression (4.44).

We make another adjustment to expressions (4.44) and (4.46) by adding

$$\sum_{1\le j\le J}\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}^0_u}\left(\Pi^j_{L_j+1}+(u+L_0-1-\tau_k)\pi^j_{L_j+1}\right)\mathbb{1}_{\{\tau_k\le u+L_0-2,\eta_k=j\}} \tag{4.47}$$

to expression (4.44) and subtracting it from expression (4.46). If an ordered unit k with $\eta_k = j$ is in $\mathcal{Q}^0_u$ where $\tau_k \le u + L_0 - 2$ and its matching demand incurs a total backorder cost of $\Pi^j_{L_0+L_j-(\tau_k-u)}$ when k arrives at retailer j, then we know that k is in the set $\mathcal{Q}^j_{u+L_0}$ because k must arrive at retailer j in period $u + L_0 + L_j$ in order for its matching demand to incur a total backorer cost of $\Pi^j_{L_0+L_j-(\tau_k-u)}$. Therefore, to expression (4.44), we can add the equivalent expression $\sum_{1\le j\le J}\sum_{u=1+L_0}^{T+1+L_0}\sum_{k\in\mathcal{Q}^j_u}\left(\Pi^j_{L_j+1}+(u-1-\tau_k)\pi^j_{L_j+1}\right)\mathbb{1}_{\{\tau_k\le u-2\}}$. Doing this, we get

$$\begin{aligned}
\sum_{1\le j\le J}\Bigg\{&\sum_{u=1}^{T+1+L_0}\sum_{k\in\mathcal{Q}^j_u}\Pi^j_{(L_j-(\tau_k-u))^+}\mathbb{1}_{\{\tau_k\ge u-1\}}\\
&+\sum_{u=1+L_0}^{T+1+L_0}\sum_{k\in\mathcal{Q}^j_u}\left(\Pi^j_{L_j+1}+(u-1-\tau_k)\pi^j_{L_j+1}\right)\mathbb{1}_{\{\tau_k\le u-2\}}\Bigg\} &(4.48)
\end{aligned}$$

for (4.44). Note that we have changed the upper limit in the outer-most summation in the first term from $T+1$ to $T+1+L_0$. This is a valid change because for values of u satisfying $T+1 < u \leq T+1+L_0$, $k \in \mathcal{Q}_u^j$ and $\tau_k \geq u-1 > T$ imply that $\tau_k = T + L_0 + L_j + 1$ since $D_{T+1} = D_{T+2} = ... = D_{T+L_0+L_j} = 0$ and $D_{T+L_0+L_j+1} = \infty$. For all these values of u, $\tau_k \geq u-1$ ensures that $(L_j-(\tau_k-u))^+ = (L_j-(T+L_0+L_j+1)+u)^+ \leq (L_j-(T+L_0+L_j+1)+T+1+L_0)^+ = 0$. Secondly, note that each summand in the second term above corresponds to the backorder cost associated with an item ordered in period u at retailer j whose matching demand arose at a time strictly before $u-1$. If we only want to keep track of the echelon inventory position at location j, there is no way to distinguish backorders of different ages that have been around the system for more than L_j+1 periods. Note, however, that the backorder costs in this term are linearized beyond L_j+1 periods. This allows us to charge $\Pi^j_{L_j+1}$ when we order a unit k at time t whose matching unit arrived at $t-1$ or before. The backorder costs beyond L_j+1 periods are incorporated by charging π^{L_1+1} per unit of negative inventory position after a procurement decision is made in every period. Therefore, after the addition of (4.47), expression (4.48) can be written as

$$\sum_{1\leq j\leq J} \sum_{u=1}^{T+1+L_0} \left(\sum_{k\in\mathcal{Q}_u^j} \Pi^j_{(L_j+u-[\tau_k\vee(u-1)])^+} + \pi^j_{L_j+1}[-r_u^j]^+ \right). \tag{4.49}$$

As for (4.46), the subtraction of (4.47) results in

$$\sum_{1\leq j\leq J} \sum_{u=1}^{T+1} \sum_{k\in\mathcal{Q}_u^0} \Psi_u^j(\tau_k) \mathbb{1}_{\{\tau_k\leq u+L_0-2, \eta_k=j\}} \tag{4.50}$$

$$\geq \sum_{u=1}^{T+1} \sum_{k\in\mathcal{Q}_u^0} \Psi_u^C(\tau_k \vee (u-1)) \tag{4.51}$$

by recognizing that $\left(\Pi^j_{(L_0+L_j-(\tau_k-u))} - \left[\Pi^j_{L_j+1} + (u+L_0-1-\tau_k)\pi^j_{L_j+1}\right]\right) =$

$\Psi_u^j(\tau_k)$ for $\tau_k \leq u + L_0 - 2$ by definition (4.8). We have now shown that

$$\begin{aligned}
&\sum_{1\leq j\leq J}\sum_{t=1}^{T}\sum_{s=1}^{M}\Pi_s^j\hat{b}_{jt}^s \\
\geq\ &\sum_{1\leq j\leq J}\sum_{u=1}^{T+1+L_0}\left(\sum_{k\in\mathcal{Q}_u^j}\Pi^j_{(L_j+u-[\tau_k\vee(u-1)])^+} + \pi^j_{L_j+1}[-r_u^j]^+\right) \\
&+\sum_{u=1}^{T+1}\sum_{k\in\mathcal{Q}_u^0}\Psi_u^C(\tau_k\vee(u-1)).
\end{aligned}$$

We now move on to the total holding cost term in (4.39):

$$\sum_{1\leq j\leq J}\sum_{t=L_j+1}^{T+L_0+L_j+1} h_j(x_t^j - D_t^j)^+. \tag{4.52}$$

Recognize first of all that the on-hand inventory at the end of t at retailer j, $(x_t^j - D_t^j)^+$, can also be expressed as $\sum_{s=1}^{t-L_j}\sum_{k\in\mathcal{Q}_s^j}\mathbb{1}_{\{\tau_k\geq t+1\}}$. Using similar tricks, we can rewrite this expression as

$$\begin{aligned}
&\sum_{1\leq j\leq J}\sum_{t=L_j+1}^{T+L_0+L_j}\sum_{s=1}^{t-L_j}\sum_{k\in\mathcal{Q}_s^j} h_j\mathbb{1}_{\{\tau_k\geq t+1\}} \\
=&\sum_{1\leq j\leq J}\sum_{s=1}^{T+L_0}\sum_{k\in\mathcal{Q}_s^j}\sum_{t=s+L_j}^{T+L_0+L_j} h_j\mathbb{1}_{\{\tau_k\geq t+1\}} \\
=&\sum_{1\leq j\leq J}\sum_{s=1}^{T+L_0+1}\sum_{k\in\mathcal{Q}_s^j} h_j\left(\tau_k-(s+L_j)\right)^+.
\end{aligned} \tag{4.53}$$

Again, the first line above follows by noting that the on-hand inventory at retailer j at the end of t corresponds to all orders placed at or before $t - L_j$ which still has not been used up by its matching demand (i.e. $\tau_k \geq t + 1$ for this unit k). The first equality follows by switching the order of summation. The last line follows simply by counting the number of non-zero terms under the inner-most summation. Note that we have also increased the upper limit of the outermost summation by 1 in the last line. This is a valid change because if $s = T + L_0 + 1$ and $k \in \mathcal{Q}_s^j$, the expression $(\tau_k - (s + L_j))^+ = (\tau_k - (T + L_0 + 1 + L_j))^+ = 0$.

This concludes the proof that our aggregate-matching cost-accounting mechanism gives rise to a lower-bound dynamic program. □

Lemma 4.4. *The definition of V_{T+L0+1} as $\sum_{1\le j\le J}\Pi^j_{L_j+1}[-I^j_{T+L_0+1}]^+$ and the definition of C_t in (4.13) imply that $C_{T+L_0+1}=0$.*

Proof: The transformation that defines C_t is:

$$C_t(I_t,z^0_t,Q^0_t)=V_t(I_t,z^t_0,Q^t_0)+\sum_{q_0=1}^{[I^0_t]^+}\mathbb{E}\{\Psi^C_t(\theta^0_t(q_0,0))\}-[-I^0_t]^+\Psi^C_t(t-1)$$
$$+\sum_{1\le j\le J}\sum_{q_j=1}^{[I^j_t]^+}E\{\Gamma^j_t(\theta^j_t(q_j,0))\}-\sum_{1\le j\le J}[-I^j_t]^+\Gamma^j_t(t-1).$$

As discussed in the chapter, in period $T+1$, we require the order-up-to level of the central warehouse to be no less than 0. Because there are no more demands after period T, I^0_t will stay non-negative until period $\hat{T}=T+L_0+1$. Therefore, $[-I^0_{\hat{T}}]^+=0$ and we have

$$\begin{aligned}C_{\hat{T}}(I_{\hat{T}},z^0_{\hat{T}},Q^0_{\hat{T}}) \quad = \quad & \sum_{1\le j\le J}\Pi^j_{L_j+1}[-I^j_{\hat{T}}]^+ +\sum_{q_0=1}^{[I^0_{\hat{T}}]^+}\mathbb{E}\{\Psi^C_{\hat{T}}(\theta^0_{\hat{T}}(q_0,0))\}\\ & +\sum_{1\le j\le J}\sum_{q_j=1}^{[I^j_{\hat{T}}]^+}E\{\Gamma^j_{\hat{T}}(\theta^j_{\hat{T}}(q_j,0))\}-\sum_{1\le j\le J}[-I^j_{\hat{T}}]^+\Gamma^j_{\hat{T}}(\hat{T}-1).\end{aligned}$$

nothing that $V_{T+L_0+1}=\sum_{1\le j\le J}\Pi^j_{L_j+1}[-I^j_{T+L_0+1}]^+$.

With the conventions $D^j_{T+1}=D^j_{T+2}=...=D^j_{T+L_0+L_j}=0$ and $D^j_{T+L_0+L_j+1}=\infty$, we have $\theta^j_{\hat{T}}(q_j,0)=T+L_0+L_j+1$ for $q_j>0$ and $0\le j\le J$. Because of this, $\mathbb{E}\{\Psi^C_{\hat{T}}(\theta^0_{\hat{T}}(q_0,0))\}=\mathbb{E}\{\Psi^C_{T+L_0+1}(T+2L_0+1)\}=0$ and $\mathbb{E}\{\Gamma^j_{\hat{T}}(\theta^j_{\hat{T}}(q_j,0))\}=\mathbb{E}\{\Gamma^j_{T+L_0+1}(T+L_0+L_j+1)\}=0$ where $q_0>0$ and $q_j>0$ using the definitions in (4.6) and (4.8). Finally, note that $\Gamma^j_{\hat{T}}(\hat{T}-1)=\Pi^j_{L_j+1}$. Because of this, the first term and last term cancel each other giving $C_{\hat{T}}=C_{T+L_0+1}=0$. □

Lemma 4.5. *As defined in (4.17) and (4.16), the immediate cost function L_t^j is convex over the $\mathbb{Z}$ for $1 \le j \le J$ and convex over $\{0, 1, 2, ...\}$ for $j = 0$.*

Proof: We refer the reader to the proof of Lemma 3.2 to see that L_t^j is convex over $\mathbb{Z}$ for $1 \le j \le J$. For $L_t^0(r_t^0)$, we use a proof analogous to the proof of Lemma 3.2. We start by conditioning on D_t^0 and writing the conditional version of L_t^0 without the linear term in r_t^0:

$$\begin{aligned}
\tilde{L}_t^0(r_t^0, D_t^0) &= L_t^0(r_t^0, D_t^0) - h_0 \mathbb{E}\left[r_t^0 - \sum_{s=t}^{t+L_0-1} D_s^0 \right] \\
&= \sum_{q_0=1}^{r_t^0} \mathbb{E}\{\Psi_t^C(\theta_t^0(q_0, 0)) \mid D_t^0\} - \sum_{q_0=1}^{[r_t^0 - D_t^0]^+} \mathbb{E}\{\Psi_{t+1}^C(\theta_{t+1}^0(q_0, 0)) \mid D_t^0\} \\
&\quad + [D_t^0 - r_t^0]^+ \, \Psi_{t+1}^C(t). \qquad (4.54)
\end{aligned}$$

First, we compute $\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0)$ when $r_t^0 > D_t^0 \ge 0$. Since $r_t^0 > D_t^0 \ge 0$, we have $\theta_t^0(r_t^0, 0) = \min\{\tau : D_t^0 + D_{t+1}^0 + \ldots + D_\tau^0 \ge r_t^0\} = \min\{\tau : D_{t+1}^0 + \ldots + D_\tau^0 \ge r_t^0 - D_t^0\} = \theta_{t+1}^0(r_t^0 - D_t^0, 0)$ by the definition of $\theta_t^0(q_0, I_t)$ in (4.5).

Therefore, if $r_t^0 > D_t^0 \ge 0$, the distributions of $\theta_t^0(r_t^0, 0)$ and $\theta_{t+1}^0(r_t^0 - D_t^0, 0)$ conditional on D_t^0 are identical. Using this fact, we get the following first difference for $r_t^0 > D_t^0$:

$$\begin{aligned}
\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0) &= \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) \mid D_t^0\} - \mathbb{E}\{\Psi_{t+1}^C(\theta_{t+1}(r_t^0 - D_t^0, 0)) \mid D_t^0\} \\
&= \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) \mid D_t^0\} - \mathbb{E}\{\Psi_{t+1}^C(\theta_t^0(r_t^0, 0)) \mid D_t^0\} \\
&= \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) - \Psi_t^C(\theta_t^0(r_t^0, 0) - 1) \mid D_t^0\}, \qquad (4.55)
\end{aligned}$$

where the second equality uses the fact that the distributions of $\theta_t^0(r_t^0, 0)$ and $\theta_{t+1}(r_t^0 - D_t^0, 0)$ conditional on D_t^0 are identical as long as $r_t^0 > D_t^0$ and the third equality uses the identity $\Psi_t^C(n) = \Psi_{t+1}^C(n+1)$ that follows from the definition of $\Psi_t^C(\cdot)$ in (4.9) and that of $\Psi_t^j(\cdot)$ in (4.8).

Second, we compute $\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0)$ when $0 < r_t^0 \leq D_t^0$ for $r_t^0 = 1, 2, \ldots$. Since $D_t^0 \geq r_t^0 \geq 1$, the definition of $\theta_t^0(q_0, I_t)$ in (4.5) implies that $\theta_t^0(r_t^0, 0) = t$. Using the definition of $\tilde{L}_t^0(r_t^0, D_t^0)$ in (4.54), if $1 \leq r_t^0 \leq D_t^0$, then the first difference $\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0)$ is given by

$$\begin{aligned}
\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0) &= \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) \,|\, D_t^0\} - \Psi_{t+1}^C(t) \\
&= \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) \,|\, D_t^0\} - \mathbb{E}\{\Psi_{t+1}^C(\theta_t^0(r_t^0, 0)) \,|\, D_t^0\} \\
&= \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) - \Psi_t^C(\theta_t^0(r_t^0, 0) - 1) \,|\, D_t^0\}, \quad (4.56)
\end{aligned}$$

where the second equality uses the fact that if $D_t^0 \geq r_t^0 \geq 1$, then $\theta_t^0(r_t^0, 0) = t$. Combining (4.55) and (4.56), we obtain $\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0) = \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) - \Psi_t^C(\theta_t^0(r_t^0, 0) - 1) \,|\, D_t^0\}$ whenever $r_t^0 > 0$, in which case, taking expectations yields $\mathbb{E}\{\tilde{L}_t^0(r_t^0, D_t^0) - \tilde{L}_t^0(r_t^0 - 1, D_t^0)\} = \mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) - \Psi_t^C(\theta_t^0(r_t^0, 0) - 1)\}$.

The definition of $\theta_t^0(q_0, I_t)$ in (4.5) implies that $\theta_t^0(r_t^0, 0)$ is stochastically increasing in r_t^0 in the sense that $\mathbb{P}\{\theta_t^0(r_t^0, 0) \geq \tau\}$ is non-decreasing in r_t^0 for all τ. Since $\Psi_t^C(\cdot)$ is convex, $\Psi_t^C(n) - \Psi_t^C(n-1)$ is non-decreasing in n. In this case, by Lemma 4.7.2 in Puterman (1994), it follows that $\mathbb{E}\{\Psi_t^C(\theta_t^0(r_t^0, 0)) - \Psi_t^C(\theta_t^0(r_t^0, 0) - 1)\}$ is non-decreasing in r_t^0 whenever $r_t^0 > 0$. That $\tilde{L}_t^0(r_t^0)$ and $L_t^0(r_t^0)$ differ only by a linear term in r_t^0 implies that L_t^0 is convex over $\{0, 1, 2, \ldots\}$. □

Lemma 4.6.

$$\min_{\substack{r_t^j \geq I_t^j, 1 \leq j \leq J \\ \sum_{1 \leq j \leq J} r_t^j \leq z_t^0}} \left\{ \sum_{1 \leq j \leq J} \left[L_t^j(r_t^j) + \mathbb{E}\{C_{t+1}^j(r_t^j - D_t^j)\} \right] \right\} \;\geq\; \Delta_t(z_t^0) \;+\; \sum_{1 \leq j \leq J} C_t^j(I_t^j)$$

Proof: This proof is based on the proof of Lemma 8 found in Kunnumkal and Topaloglu (2008). Notice all the terms on the right hand side are defined by a minimization problem. Recall that

$$C_t^j(I_t^j) = \min_{r_t^j \geq I_t^j} \left\{ L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j) \right\} \tag{4.57}$$

and

$$\begin{aligned}\Delta_t(z_t^0) = \min_{\sum_{1\leq j\leq J} r_t^j \leq z_t^0} \Bigg\{ \sum_{j=1}^{J} \big\{ L_t^j(r_t^j) + \mathbb{E}C_{t+1}^j(r_t^j - D_t^j) \\ -L_t^j(r_t^{j*}) + \mathbb{E}C_{t+1}^j(r_t^{j*} - D_t^j) \big\} \Bigg\}\end{aligned} \tag{4.58}$$

where r_t^{j*} is the unconstrained minimizer for problem (4.57). Suppose that the optimal solution for the problem on the left hand side of the inequality in the lemma is given by $\hat{r}_t^j$ for $1 \leq J$, we proceed by constructing feasible solutions for the minimization problems that define the right hand side. In particular, consider the feasible solution $\tilde{r}_t^j = r_t^{j*} \wedge \hat{r}_t^j$ for the minimization problem which defines Δ_t. (We know that this solution is feasible because $\hat{r}_t^j$, $1 \leq j \leq J$ satisfy the constraint $\sum_{1\leq j\leq J} \hat{r}_t^j \leq z_t^0$ as the optimal solution for the problem on the left hand side of the inequality given in the lemma and $\tilde{r}_t^j \leq \hat{r}_t^j$ for $1 \leq j \leq J$.) The feasibility of $\tilde{r}_t^j$ implies that

$$\begin{aligned}\Delta_t(z_t^0) &\leq \Bigg\{ \sum_{j=1}^{J} \big\{ L_t^j(\tilde{r}_t^j) + \mathbb{E}C_{t+1}^j(\tilde{r}_t^j - D_t^j) - L_t^j(r_t^{j*}) + \mathbb{E}C_{t+1}^j(r_t^{j*} - D_t^j) \big\} \Bigg\} \\ &= \Bigg\{ \sum_{j=1}^{J} \mathbb{1}_{\{\hat{r}_t^j \leq r_t^{j*}\}} \big\{ L_t^j(\hat{r}_t^j) + \mathbb{E}C_{t+1}^j(\hat{r}_t^j - D_t^j) \\ &\qquad -L_t^j(r_t^{j*}) + \mathbb{E}C_{t+1}^j(r_t^{j*} - D_t^j) \big\} \Bigg\}.\end{aligned} \tag{4.59}$$

The equality follows by noting that $\tilde{r}_t^j = r_t^{j*}$ if $\hat{r}_t^j > r_t^{j*}$. Otherwise, we have $\tilde{r}_t^j = \hat{r}_t^j$ if $\hat{r}_t^j \leq r_t^{j*}$.

On the other hand, problem (4.57) is solved by r_t^{j*} if $I_t^j \leq \hat{r}_t^j \leq r_t^{j*}$. Otherwise,

$\hat{r}_t^j \geq I_t^j$ is a feasible solution and hence,

$$\begin{aligned} C_t^j(I_t^j) \leq \quad & \mathbb{1}_{\left\{\hat{r}_t^j \leq r_t^{j*}\right\}} \left\{L_t^j(r_t^{j*}) + \mathbb{E}C_{t+1}^j(r_t^{j*} - D_t^j)\right\} \\ & + \mathbb{1}_{\left\{\hat{r}_t^j > r_t^{j*}\right\}} \left\{L_t^j(\hat{r}_t^j) + \mathbb{E}C_{t+1}^j(\hat{r}_t^j - D_t^j)\right\}. \end{aligned} \tag{4.60}$$

Adding the right-hand side of (4.59) to the right-hand side of (4.60) summed over $1 \leq j \leq J$, we get $\sum_{1 \leq j \leq J} \left[L_t^j(\hat{r}_t^j) + \mathbb{E}\{C_{t+1}^j(\hat{r}_t^j - D_t^j)\}\right]$ which is precisely the left-hand side of the inequality in the lemma. $\square$

BIBLIOGRAPHY

D. Adelman and A. J. Mersereau. Relaxations of weakly coupled stochastic dynamic programs. *Operations Research*, 56(3):712–727, 2008.

M. Al-Gwaiz, S. Huber, and M. F. M. Lee. Optimizing Service Parts Inventory For Honeywell Aerospace Assuming Correlated Failures. *Master's Thesis, Cornell University*, 2006.

S. Axsater. Simple solution procedures for a class of two-echelon inventory problems. *Operations Research*, 38(1):64–69, 1990.

D. P. Bertsekas and J. N. Tsitsiklis. *Neuro-Dynamic Programming*. Athena Scientific, 1996.

J.R. Birge and F. Louveaux. *Introduction to stochastic programming*. Springer Verlag, 1997.

F. Cheng, M. Ettl, G. Lin, and D. D. Yao. Inventory-service optimization in configure-to-order systems. *Manufacturing & Service Operations Management*, 4(2):114–132, 2002.

A. J. Clark and H. Scarf. Optimal policies for a multi-echelon inventory problem. *Management science*, 6(4):475–490, 1960.

M. K. Dogru, M. I. Reiman, and Q. Wang. A Stochastic Programming Based Inventory Policy for Assemble-to-Order Systems with Application to the W Model. *Operations Research*, 2010.

G. Eppen and L. Schrage. Centralized ordering policies in a multi-warehouse system with lead times and random demand. *Multi-level production/inventory control systems: Theory and practice*, 16:51–67, 1981.

A. Erdelyi and H. Topaloglu. Approximate Dynamic Programming for Dynamic Capacity Allocation. *Forthcoming*, 2009.

A. Federgruen and P. Zipkin. Approximations of dynamic, multilocation production and inventory problems. *Management Science*, 30(1):69–84, 1984.

Y. Gerchak and M. Henig. Component commonality in assemble-to-order systems: Models and properties. *Naval Research Logistics*, 36(1):61–68, 2006.

Y. Gerchak, M. J. Magazine, and A. B. Gamble. Component commonality with service level requirements. *Management science*, 34(6):753, 1988.

P. Glasserman and Y. Wang. Leadtime-inventory trade-offs in assemble-to-order systems. *Operations Research*, 46(6):858–871, 1998.

K.S. Goyal and B.C. Giri. Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, 134(1):1–16, 2001.

V. D. R. Guide et al. Repairable inventory theory: models and applications. *European Journal of Operational Research*, 102(1):1–20, 1997.

W. H. Hausman, H. L. Lee, and A. X. Zhang. Joint demand fulfillment probability in a multi-item inventory system with independent order-up-to policies. *European Journal of Operational Research*, 109(3):646–659, 1998.

W.T. Huh, G. Janakiraman, A. Muharremoglu, and A. Sheopuri. Inventory systems with a generalized cost model. *Operations Research*, Forthcoming, 2011.

W. J. Kennedy, J. Wayne Patterson, and L. D. Fredendall. An overview of recent literature on spare parts inventories. *International Journal of production economics*, 76(2):201–215, 2002.

S. Kunnumkal and H. Topaloglu. A duality-based relaxation and decomposition approach for inventory distribution systems. *Naval Research Logistics*, 55(7): 612–631, 2008.

S. Kunnumkal and H. Topaloglu. A new dynamic programming decomposition method for the network revenue management problem with customer choice behavior. *Production and Operations Management*, To appear, 2009.

R. Levi, M. Pal, R.O. Roundy, and D.B. Shmoys. Approximation algorithms for stochastic inventory control models. *Mathematics of Operations Research*, 32(2): 284–302, 2007.

Y. Lu and J. S. Song. Order-based cost optimization in assemble-to-order systems. *Operations Research*, 53(1):151, 2005.

Y. Lu, J. S. Song, and D. D. Yao. Order fill rate, leadtime variability, and advance demand information in an assemble-to-order system. *Operations Research*, 51 (2):292–308, 2003.

Y. Lu, J. S. Song, and D. D. Yao. Backorder minimization in multiproduct assemble-to-order systems. *IIE Transactions*, 37(8):763–774, 2005.

Y. Lu, J. S. Song, and Y. Zhao. No-Holdback Allocation Rules for Continuous-Time Assemble-to-Order Systems. *Operations Research*, 2010.

J. Mamer and S. Smith. Inventories for sequences of multi-item demands. *Supply Chain Structures: Coordination, Information and Optimization, J.S. Song, D.D. Yao (eds.)*, pages 415–437, 2001.

J. A. Muckstadt. *Analysis and algorithms for service parts supply chains*. Springer Verlag, 2005.

A. Muharremoglu and J.N. Tsitsiklis. A single-unit decomposition approach to multi-echelon inventory systems. *Operations Research*, 56(5):1089–1103, 2008.

S. Nahmias. Managing repairable item inventory systems: a review. *Multi-Level Production/Inventory Control Systems: Theory and Practice*, 16:253–277, 1981.

S. Nahmias. Perishable inventory theory: A review. *Operations Research*, 30(4): 680–708, 1982.

S. Nahmias. *Perishable Inventory Systems*. Springer, New York, NY, 2011.

E.L. Porteus. *Foundations of Stochastic Inventory Theory*. Stanford University Press, Stanford, CA, 2002.

W. B. Powell. *Approximate Dynamic Programming: Solving the curses of dimensionality*. Wiley-Blackwell, 2007.

M. L. Puterman. *Markov Decision Processes*. John Wiley and Sons, Inc., New York, 1994.

K. Rosling. Optimal inventory policies for assembly systems under random demands. *Operations Research*, pages 565–579, 1989.

K. Rosling. Inventory cost rate functions with nonlinear shortage costs. *Operations Research*, 50(6):1007–1017, 2002.

J. S. Song. On the order fill rate in a multi-item, base-stock inventory system. *Operations Research*, 46(6):831–845, 1998.

J. S. Song. A note on assemble-to-order systems with batch ordering. *Management Science*, 46(5):739–743, 2000.

J. S. Song. Order-based backorders and their implications in multi-item inventory systems. *Management Science*, 48(4):499–516, 2002.

J. S. Song and D. D. Yao. Performance analysis and optimization of assemble-to-order systems with random lead times. *Operations Research*, 50(5):889–903, 2002.

J. S. Song and P. Zipkin. Supply chain operations: Assemble-to-order systems. *Handbooks in Operations Research and Management Science*, 11:561–596, 2003.

J. S. Song, S. H. Xu, and B. Liu. Order-fulfillment performance measures in an assemble-to-order system with stochastic leadtimes. *Operations Research*, 47 (1):131–149, 1999.

H. Topaloglu. Using Lagrangian relaxation to compute capacity-dependent bid-prices in network revenue management. *Operations research*, 57(3):637–649, 2009.

J. A. Van Mieghem and N. Rudi. Newsvendor networks: Inventory management and capacity investment with discretionary activities. *Manufacturing and Service Operations Management*, 4(4):313–335, 2002.

I. Vliegen. Integrated Planning for Service Tools and Spare Parts for Capital Goods. *Ph.D. Thesis, Eindhoven University of Technology*, 2009.

S. W. Yap, K. Lau, and M. Kornfield. Optimizing Spare Parts Inventory For Honeywell Aerospace Assuming Correlated Failures. *Master's Thesis, Cornell University*, 2005.

A. X. Zhang. Demand Fulfillment Rates in an Assemble-to-Order System with Multiple Products and Dependent Demands. *Production and Operations Management*, 6(3):309–324, 2009.

P.H. Zipkin. *Foundations of Inventory Management*. McGraw-Hill, Boston, MA, 2000.